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DETERMINATION OF RESIDUAL STRESSES IN COATED PLATES USING LAYER GROWING/REMOVING METHODS: 100TH ANNIVERSARY OF STONEY'S EQUATIONS

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Abstract: *In connection with the 100th anniversary of Stoney's equations, some historical remarks are made with respect to the development of these equations. In the capacity of an example of the extension of Stoney's equations, a unique algorithm of the layer growing/removing methods for determination of residual stresses in isotropic inhomogeneous coated plates is presented. Using a computer program based on this algorithm, residual stresses are computed in a detonation gun sprayed WC-Co coating.*

1. INTRODUCTION

The layer growing method (non-destructive method) and the layer removing method (destructive method) are used for experimental determination of residual stresses in coated plates. Elaboration of the theory of the layer growing method started 100 years ago with a seminal paper by Stoney [1] and was evolved in studies by Brenner and Senderoff [2], by Kõo [3], by Davidenkov [4], by Birger and Kozlov [5], by Doi et al. [6] as well as by other authors. The theory of the layer removing method was evolved in papers by Stäblein [7], Treuting and Read [8] and Moore and Evans [9], who treated homogeneous plates. Unique consideration of the layer growing and layer removing methods started with paper [10] presenting the general principles of this approach.

In paper [11] a common algorithm of the layer growing and layer removing methods is presented for determination of biaxial residual stresses in a free rectangular orthotropic inhomogeneous elastic plate whose elastic parameters depend on its thickness coordinate continuously or piecewise. This algorithm enables to calculate residual stresses from the curvature or strain measured on the stationary surface of the plate (substrate), as well as from initial stresses measured on the moving surface by the X-ray diffraction technique. In this study a special algorithm that follows from the general algorithm for an isotropic inhomogeneous coated plate is considered and applied for determination of residual stresses in a detonation gun sprayed WC-Co coating from the curvatures according to the data by Wang et al. [12].

2. SOME HISTORICAL REMARKS RELATING TO STONEY'S EQUATIONS

In 1909 G. Gerald Stoney published his famous paper "The Tension of Metallic Films deposited by Electrolysis" [1]. In this paper he derived three equations for experimental determination of residual stresses originating from a galvanic coating at application onto one side of a steel strip from the curvature of the strip measured after deposition.

The premier equation, known as Stoney's formula, follows from the equilibrium conditions of the coated strip and read as:

$$\bar{\sigma} = \frac{E h_1^2}{6 h_2 r}, \qquad (1)$$

where $\bar{\sigma}$ is residual (initial) stress in the coating, E is the modulus of elasticity

(Young's modulus) of the substrate, h_1 is the thickness of the substrate, h_2 is the thickness of the coating and r is the radius of the curvature of the initially flat substrate after deposition of the coating.

The second equation, which represents Stoney's formula in a differential form and follows from equation (1), is:

$$\bar{\sigma} = \frac{E\left(h_1 + y\right)^2}{6} \frac{\mathrm{d}æ}{\mathrm{d}y}, \qquad (2)$$

where y is the current thickness of the coating and $æ = 1/r$ is the curvature of the substrate.

The third equation, resulting from differential expression (2) after integration provided that stress $\bar{\sigma}$ remains constant with thickness, is

$$\bar{\sigma} = \frac{E\left(h_1 + h_2\right)h_1}{6h_2} æ, \qquad (3)$$

where h_2 is the final thickness of the coating.

In 1949 Brenner and Senderoff [2] extended Stoney's equations for the case of thick coatings of strip substrates with various boundary conditions and with the modulus of elasticity differing from that of the coating.

In particular, they considered the substrate with slipping ends and developed a theory which allows determination of residual stress σ in the coating as the sum of initial (instantaneous) stress $\bar{\sigma}$ and additional stress σ^*:

$$\sigma = \bar{\sigma} + \sigma^*. \qquad (4)$$

It is worth mentioning that Stoney, Brenner and Senderoff treated the coating process as a consecutive application of elementary layers equidistant from the substrate surface, i.e. used model that is known as the model of continuous growth in layers [10].

The above analyses were based on the same assumption, i.e. the state of stress of the coated substrate is uniaxial. In 1959 Kõo [3] and in 1960 Davidenkov [4] pointed out that the stress state of a substrate with coating is biaxial but not uniaxial as was assumed previously; they showed that the difference can be taken into account by multiplying the stress values, obtained from equations (1) – (3), by factor $1/(1 - \mu)$, where μ is the Poisson's ratio of the substrate material.

Thus, today Stoney's equations (1) and (2) are generally used in the form:

$$\bar{\sigma} = \frac{E}{1-\mu} \frac{h_1^2}{6h_2} æ; \qquad (5)$$

$$\bar{\sigma} = \frac{E}{1-\mu} \frac{\left(h_1 + y\right)^2}{6} \frac{\mathrm{d}æ}{\mathrm{d}y}, \qquad (6)$$

where the factor $E/(1 - \mu)$ is known as the biaxial modulus of the substrate material.

It is possible to express curvature æ by means of the strain ε measured by a strain gauge on the free stationary surface of the substrate. Then, instead of expressions (5) and (6), one has [13]

$$\bar{\sigma} = \frac{E}{1-\mu} \frac{h_1}{2h_2} \varepsilon; \qquad (7)$$

$$\bar{\sigma} = \frac{E}{1-\mu} \frac{h_1 + y}{2} \frac{\mathrm{d}\varepsilon}{\mathrm{d}y}. \qquad (8)$$

Today Stoney's equations (5) and (6) and their extensions are frequently used for relating the substrate's curvature to coating stress. In particular, equation (5) serves as a basis for standard [14].

3. AN ALGORITHM OF LAYER GROWING/REMOVING METHODS FOR PLATES

Following general algorithm [11] consider a thin layer growing on one face of a free rectangular plate (Fig. 1).

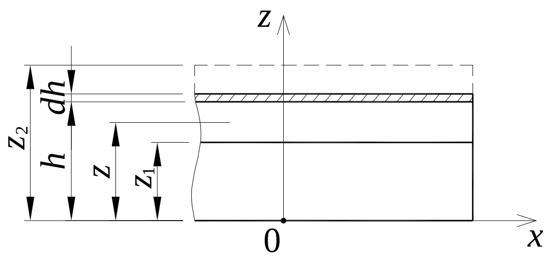

Fig. 1. Layer-growing on the upper face of a rectangular plate

Let the initial thickness of the plate be z_1, variable thickness h and final thickness z_2.

Rectangular coordinates *x*, *y* and *z* are used, where the free stationary surface of the plate is taken as the reference surface (*x*, *y*) and coordinate *z* is perpendicular to the stationary surface. It is assumed that axes *x* and *y* are both orthotropic axes and principal axes of the state of residual stresses depending on coordinate *z* only. It is also assumed that the edges of the plate are parallel to axes *x* and *y*.

According to the general algorithm, residual stresses in layer *z* of the coating can be calculated as the sum of initial and additional stresses:

$$\{\sigma\} = \{\bar{\sigma}\} + \{\sigma^*\}, \tag{9}$$

where

$$\{\sigma\} = \begin{bmatrix} \sigma_x & \sigma_y \end{bmatrix}^{\mathrm{T}},$$

$$\{\bar{\sigma}\} = \begin{bmatrix} \bar{\sigma}_x & \bar{\sigma}_y \end{bmatrix}^{\mathrm{T}},$$

$$\{\sigma^*\} = \begin{bmatrix} \sigma_x^* & \sigma_y^* \end{bmatrix}^{\mathrm{T}}$$

are the vectors of residual stresses, initial stresses and additional stresses, respectively.

Initial stresses $\bar{\sigma}_x = \bar{\sigma}_x(h),\ \ \bar{\sigma}_y = \bar{\sigma}_y(h)$ in differential surface layer d*h* can be expressed by strains $\varepsilon_x = \varepsilon_x(h),\ \ \varepsilon_y = \varepsilon_y(h),$ and curvatures $æ_x = æ_x(h),\ \ æ_y = æ_y(h)$ measured on the stationary surface as follows:

$$\{\bar{\sigma}\} = [B]\left\{\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}h}\right\} - [C]\left\{\frac{\mathrm{d}\tilde{æ}}{\mathrm{d}h}\right\}; \tag{10}$$

$$h\{\bar{\sigma}\} = [C]\left\{\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}h}\right\} - [D]\left\{\frac{\mathrm{d}\tilde{æ}}{\mathrm{d}h}\right\}, \tag{11}$$

where

$$\left\{\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}h}\right\} = \begin{bmatrix} \dfrac{\mathrm{d}\tilde{\varepsilon}_x}{\mathrm{d}h} & \dfrac{\mathrm{d}\tilde{\varepsilon}_y}{\mathrm{d}h} \end{bmatrix}^{\mathrm{T}}$$

is the vector of the derivatives of strain changes

$$\tilde{\varepsilon}_x = \varepsilon_x(z_2) - \varepsilon_x(h),\ \tilde{\varepsilon}_y = \varepsilon_y(z_2) - \varepsilon_y(h),$$

$$\left\{\frac{\mathrm{d}\tilde{æ}}{\mathrm{d}h}\right\} = \begin{bmatrix} \dfrac{\mathrm{d}\tilde{æ}_x}{\mathrm{d}h} & \dfrac{\mathrm{d}\tilde{æ}_y}{\mathrm{d}h} \end{bmatrix}^{\mathrm{T}}$$

is the vector of the derivatives of curvature changes

$$\tilde{æ}_x = æ_x(z_2) - æ_x(h),$$

$$\tilde{æ}_y = æ_y(z_2) - æ_y(h),$$

and

$$[B] = \begin{bmatrix} B & B_\mu \\ B_\mu & B \end{bmatrix},\ [C] = \begin{bmatrix} C & C_\mu \\ C_\mu & C \end{bmatrix},$$

$$[D] = \begin{bmatrix} D & D_\mu \\ D_\mu & D \end{bmatrix} \tag{12}$$

are the matrices of the elastic parameters given by

$$\begin{bmatrix} B & B_\mu \\ C & C_\mu \\ D & D_\mu \end{bmatrix} = \int_0^h \begin{bmatrix} E^0(z) \end{bmatrix} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} \mathrm{d}z \tag{13}$$

with

$$\begin{bmatrix} E^0 \end{bmatrix} = \begin{bmatrix} E^0 & E_\mu^0 \end{bmatrix} \tag{14}$$

and

$$E^0 = \frac{E}{1-\mu^2},\ E_\mu^0 = \mu E^0, \tag{15}$$

where $E = E(z)$ denotes the modulus of elasticity, and $\mu = \mu(z)$ denotes Poisson's ratio.

The expression for computing additional stresses in the coating $(z_1 \le z \le z_2)$ is

$$\{\sigma^*\} = \begin{bmatrix} E^* \end{bmatrix} \int_z^{z_2} \begin{bmatrix} -\left\{\dfrac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}h}\right\} & z\left\{\dfrac{\mathrm{d}\tilde{æ}}{\mathrm{d}h}\right\} \end{bmatrix} \mathrm{d}h, \tag{16}$$

where

$$\begin{bmatrix} E^* \end{bmatrix} = E^0(z)\begin{bmatrix} 1 & \mu(z) \\ \mu(z) & 1 \end{bmatrix}. \tag{17}$$

For computing residual stresses in the substrate $(0 \le z \le z_1)$, the lower limit *z* of the integral in expression (16) should be replaced by z_1.

If measurement of strains or curvatures is not performed during coating growth then, using the removing procedure, it should be assumed in the above algorithm that

$$\varepsilon_x(z_2) = \varepsilon_y(z_2) = æ_x(z_2) = æ_y(z_2) = 0.$$

Expressions (9) - (17) form a common algorithm of the layer growing/removing methods for isotropic inhomogeneous plates, allowing calculation of residual stresses at growing/removing on one face of the plate:

1. From strains and curvatures measured on the free stationary surface $(z=0)$ depending on thickness h. In this case initial stresses are computed by using expression (10) or (11). From equation (16) the expression for computing additional stresses is

$$\left\{\sigma^{*}\right\}=\left[E^{*}\right]\left[\{\tilde{\varepsilon}\} \quad -z\{\tilde{æ}\}\right]. \qquad (18)$$

2. From measured strains or curvatures only. In this case the unmeasured deformation parameter is computed from the equation

$$\left[[C]-h[B]\right]\left\{\frac{\mathrm{d}\,\tilde{\varepsilon}}{\mathrm{d}\,h}\right\}=$$

$$=\left[[D]-h[C]\right]\left\{\frac{\mathrm{d}\,\tilde{æ}}{\mathrm{d}\,h}\right\}, \qquad (19)$$

which follows from expressions (10) and (11).

3. From initial stresses measured by the X-ray diffraction technique on the moving surface depending on thickness h. In this case the derivatives of strain and curvature changes are calculated from the expressions:

$$\left\{\frac{\mathrm{d}\tilde{æ}}{\mathrm{d}h}\right\}=\frac{[C]-h[B]}{[B][D]-[C]^{2}}\{\bar{\sigma}\}; \qquad (20)$$

$$\left\{\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}h}\right\}=\frac{[D]-h[C]}{[B][D]-[C]^{2}}\{\bar{\sigma}\}, \qquad (21)$$

which are obtained by solving equations (10) and (11).

By solving equations (20) and (21) with respect to the vector of initial stresses one obtains the expressions

$$\{\bar{\sigma}\}=\frac{[B][D]-[C]^{2}}{[C]-h[B]}\left\{\frac{\mathrm{d}\tilde{æ}}{\mathrm{d}h}\right\}; \qquad (22)$$

$$\{\bar{\sigma}\}=\frac{[B][D]-[C]^{2}}{[D]-h[C]}\left\{\frac{\mathrm{d}\tilde{\varepsilon}}{\mathrm{d}h}\right\}, \qquad (23)$$

which present the extensions of Stoney's equations (6) and (8) for a biaxial stress state and for an inhomogeneous substrate-coating system.

4. COMPUTER PROGRAM RS-PLATE AND A COMPUTATION EXAMPLE

On the basis of the presented algorithm (Sect. 3), a computer program RS-PLATE is developed, which enables calculation of residual stresses in isotropic inhomogeneous plates from strains, curvatures or initial stresses measured during the growing or removing process. According to the program, calculation of the derivatives of experimental data is carried out by a preliminary fitting with a polynomial.

As an example, the program RS-PLATE is used for computing exibiaxial residual stresses $\sigma_x=\sigma_y=\sigma$ in a detonation gun sprayed WC-Co coating from the curvatures according to the data by Wang et al. [12] after formation of coatings with different thicknesses on flat stainless steel substrates wit a size of 60×18×3 mm. Fig. 2 presents the data of curvatures and the distribution of initial and residual stresses in the WC-Co coating [z_2 = 3.73 mm; μ_2 = = 0.25; $E_2(z)$ = 220 (3.05), 234 (3.10), 244 (3.15), 250 (3.20), 254 (3.25), 260 (3.30), 264 (3.35), 276 (3.40), 283 (3.45), 291 (3.50), 302 (3.55) GPa (mm)] on a steel substrate (z_1 = 3.00 mm; μ_1 = 0.30; E_1 = = 198 GPa).

Fig. 2 shows also residual stresses calculated by Wang et al. by means of commonly used Stoney's formula extended by introduction of a stress correction coefficient for thick coatings. It is evident that residual stresses by Wang et al. differ in some degree from those calculated with the use of the program RS-PLATE.

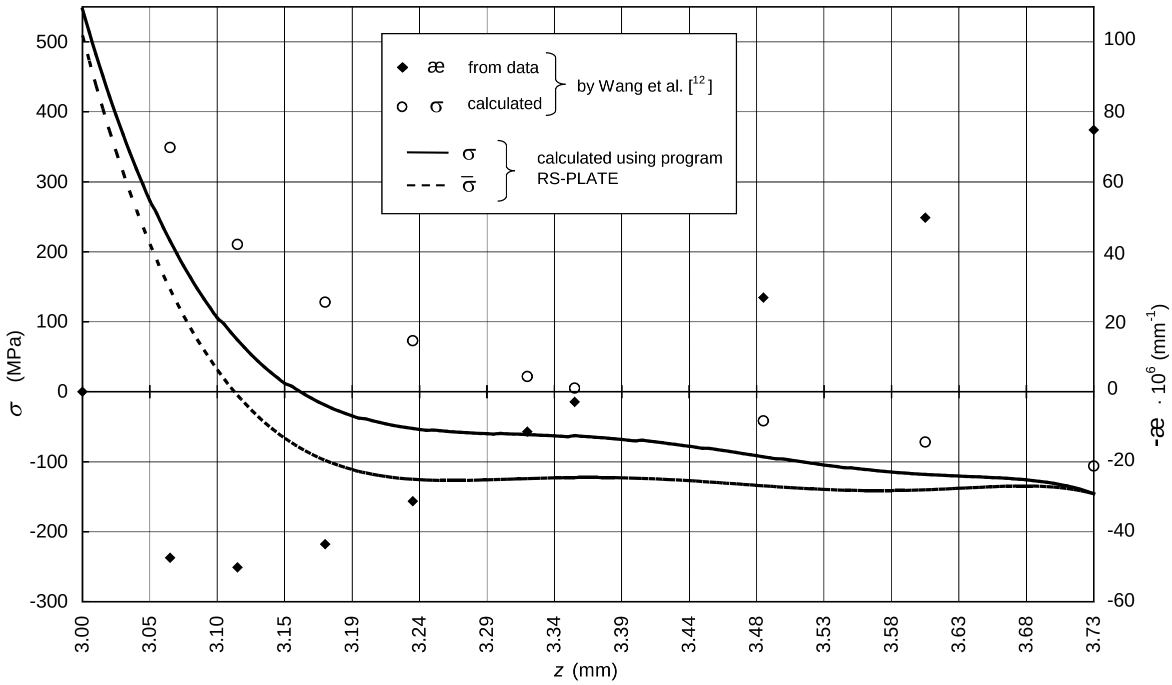

Fig. 2. Experimental data and distribution of initial and residual stresses in a detonation gun sprayed WC-Co coating on a flat stainless steel plate

5. CONCLUSIONS

1. In connection with the 100th anniversary of Stoney's equations, some historical remarks are made with respect to the development of these equations.
2. An algorithm arising from a more common algorithm [11] is presented for calculation of residual stresses in a free rectangular isotropic plate inhomogeneous along thickness. The algorithm is universal and allows calculation of residual stresses at layer growing or layer removing from strains or curvatures measured on the stationary surface, or from initial stresses measured on the moving surface of the plate.
3. Stoney's equations for a thick coating are extended for the biaxial stress state and for a substrate-coating system inhomogeneous along thickness.
4. The computer program RS-PLATE based on the presented algorithm is introduced.
5. Using the program RS-PLATE residual stresses are computed in a detonation gun sprayed WC-Co coating from curvatures according to the data by Wang et al. [12].

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