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HOLONOMIC AND NONHOLONOMIC CONSTRAINTS IN DYNAMICS OF MULTIBODY SYSTEMS

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Abstract: *This paper deals with motion of rigid bodies with articulation joints, and motion of tethered bodies. The general problem of system kinematics is presented in the first part and the motion of rigid bodies with constraints in the second part. The kinematics of the system is solved for constraints expressed in terms of coordinates and constraints expressed in terms of velocities. The relative motion of systems in central gravitational field with respect to a moving reference frame whose origin is on a circular orbit is presented.*
Key words: constraints, dynamics, multi-body system

1. INTRODUCTION

When the motion of a system of bodies which compose a large orbital station is described with respect to reference frames having origin in the centre of the attractive body (Earth) the problem of integration of motion equations presents some difficulties, because some coordinates (like vector radii) have very great values, and others (like distances between bodies) have very small values. Some difficulties can be avoided if relative motion of the system is studied with respect to a reference frame with known motion. Relative motion study isn't imposed by integration considerations but by practical aspects.

This paper studies multi-body system motion using Lagrange equations for holonomic and nonholonomic systems.

Translation conditions and rotation conditions are analyzed. In the case of rigid body motion equation of mass centre are completed with motion equation of rotation with respect to mass centre. The two kinds of equations (of mass centre and of rotation with respect to mass centre) can be parted only in particular cases. The kinematics of body systems is solved using coupling mechanism analysis under the aspect of degrees-of-freedom. The motion in central gravitational field is studied with respect a movable reference frame with origin on a circular orbit. The problem of multi-body system dynamics is solved using Lagrange equations of motion with multipliers and constraints. For the system of two tethered bodies obtained results by integration of motion equations are presented. The models and the elaborated method allow solving a large number of multi-body systems dynamics problems.

2. KINEMATICS OF SYSTEMS OF RIGID BODIES

2.1 General problem

Let two bodies (i) and (j) be with constrained motions by a coupling mechanism which is made precise by points O_i, O_j (Fig.1). The motion of the body (i) with respect the inertial reference frame $O_{0x0y0z0}$ is determined by position vector of mass centre $\overline{O_0C_i}$ and by matrix $\left[A_{i0}\right]$ which gives the attitude of $C_{ixiyizi}$ trihedral, jointed with (i) body, with respect $O_{0x0y0z0}$ reference frame. In the same way are defined position vector $\overline{O_0C_j}$ and matrix $\left[A_{j0}\right]$ for the body (j). Each body, (i) or (j), has 6 degrees-of-freedom, when it is a free body. The number of degrees-of-freedom is reduced by the number of constrains which are imposed by coupling mechanism. If the general motion of bodies (i) and (j) with respect the inertial reference frame $O_{0x0y0z0}$ are known, then the relative motion of the body (i) with respect (j) can be determined by the vector

$$\overline{O_jO_i} = \left(\overline{O_0C_i} + \overline{C_iO_i}\right) - \left(\overline{O_0C_j} + \overline{C_jO_j}\right) \tag{1}$$

and by matrix $\lfloor A_{ij} \rfloor$ which gives the attitude of (i) body with respect (j) body,

$$[A_{ij}] = [A_{i0}][A_{j0}]^T \tag{2}$$

The matrix $[A_{i0}]$ allows expressing unit vectors of $C_{ixiyizi}$ trihedral with respect unit vectors of $O_{0x0y0z0}$ trihedral,

$$\begin{Bmatrix} \overline{i_i} \\ \overline{j_i} \\ \overline{k_i} \end{Bmatrix} = [A_{i0}] \begin{Bmatrix} \overline{i_0} \\ \overline{j_0} \\ \overline{k_0} \end{Bmatrix} \tag{3}$$

For unit vectors of $C_{ixiyizi}$ trihedral the bellow relation can be written,

$$\begin{Bmatrix} \overline{i_j} \\ \overline{j_j} \\ \overline{k_j} \end{Bmatrix} = [A_{j0}] \begin{Bmatrix} \overline{i_0} \\ \overline{j_0} \\ \overline{k_0} \end{Bmatrix} \tag{4}$$

The attitude of (i) body with respect to (j) body is given by matrix $\lfloor A_{ij} \rfloor$, with relations

$$\begin{Bmatrix} \overline{i_i} \\ \overline{j_i} \\ \overline{k_i} \end{Bmatrix} = [A_{ij}] \begin{Bmatrix} \overline{i_j} \\ \overline{j_j} \\ \overline{k_j} \end{Bmatrix} \tag{5}$$

and the attitude of (j) body with respect (i) is given by $\lfloor A_{ji} \rfloor$ matrix from relations

$$\begin{Bmatrix} \overline{i_j} \\ \overline{j_j} \\ \overline{k_j} \end{Bmatrix} = [A_{ji}] \begin{Bmatrix} \overline{i_i} \\ \overline{j_i} \\ \overline{k_i} \end{Bmatrix} \tag{6}$$

From relations (5) and (6) it follows

$$[A_{ij}] = [A_{ji}]^T \tag{7}$$

If relation (3) is multiplied to the left with $[A_{i0}]^T$ and (4) is multiplied to the left with $[A_{j0}]^T$

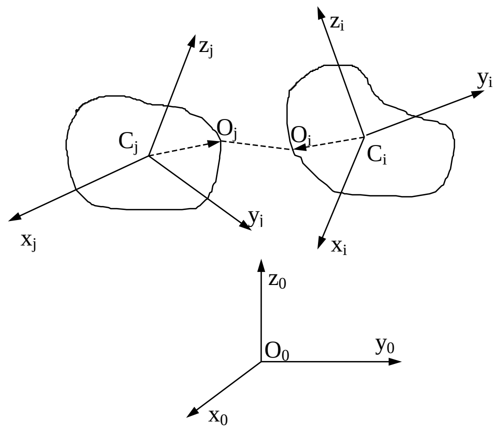

Fig. 1: System of rigid bodies

and the obtained results are compared, the equality

$$[A_{i0}]^T \begin{Bmatrix} \overline{i_i} \\ \overline{j_i} \\ \overline{k_i} \end{Bmatrix} = [A_{j0}]^T \begin{Bmatrix} \overline{i_j} \\ \overline{j_j} \\ \overline{k_j} \end{Bmatrix} \tag{8}$$

results, from where, by multiplying to the left with $[A_{i0}]$, we obtain

$$\begin{Bmatrix} \overline{i_i} \\ \overline{j_i} \\ \overline{k_i} \end{Bmatrix} = [A_{i0}][A_{j0}]^T \begin{Bmatrix} \overline{i_j} \\ \overline{j_j} \\ \overline{k_j} \end{Bmatrix} \tag{9}$$

From (5) and (9) relation (2) is obtained, which is used to compute the matrix $\lfloor A_{ij} \rfloor$, if the matrices $[A_{i0}]$ and $\lfloor A_{j0} \rfloor$ are known. Terms of $[A_{i0}]$ and $\lfloor A_{j0} \rfloor$ matrices depend of attitude angles of (i) body and (j) body with respect to inertial reference frame. If the orientation of (i) body with respect to inertial reference frame is made precise by φ_{1i}, φ_{2i}, φ_{3i} angles, which correspond to the 1-2-3 sequence of rotations with respect to a parallel reference frame with inertial reference frame $O_{0x0y0z0}$, than the bellow matrix ([8])

$$[A_{i0}] = \begin{bmatrix} c_{2i}c_{3i} & s_{1i}s_{2i}c_{3i} + s_{3i}c_{1i} & -c_{1i}s_{2i}c_{3i} + s_{1i}s_{3i} \\ -c_{2i}s_{3i} & -s_{1i}s_{2i}s_{3i} + c_{1i}c_{3i} & c_{1i}s_{2i}s_{3i} + s_{1i}c_{3i} \\ s_{2i} & -s_{1i}c_{2i} & c_{1i}c_{2i} \end{bmatrix} \tag{10}$$

and angular velocity

$$\{\omega_{i0}\} = \begin{bmatrix} c_{2i}c_{3i} & s_{3i} & 0 \\ -c_{2i}s_{3i} & c_{3i} & 0 \\ s_{2i} & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\varphi}_{1i} \\ \dot{\varphi}_{2i} \\ \dot{\varphi}_{3i} \end{Bmatrix} \tag{11}$$

are obtained. In the above relations notations of the following form were used:

$$s_{1i} = \sin\varphi_{1i}, \quad c_{1i} = \cos\varphi_{1i} \qquad (12)$$

2.2 Constraints expressed by coordinates

When constraints are functions of coordinates the motion of systems of rigid bodies can be studied with Lagrange equations for holonomic systems with dependent variables. Coupling mechanisms between body (i) and body (j) imposes restrictions on relative motion of the body. Bellow, some simple coupling mechanisms for which constraints can be expressed with functions of coordinates are analyzed.

2.2.1. Free linkage

When the coupling mechanism doesn't impose restrictions coordinates which are describing relative motion (displacements and rotations) number of constraints is zero. Each body has 6 degrees-of-freedom and the motion is studied considering two free bodies, despite of the coupling mechanism, which permits, translations with respect three directions and rotations about three axes. The case of "free" linkage is a limit case and it presents the importance only for the case in which a particular coupling mechanism becomes a "free" linkage. As an example can be considered the case of tethered bodies for particular situation of zero tension in the cable.

2.2.2. Fixed linkage

When the relative motion of (i) body with respect to (j) body has zero degrees-of-freedom the system of two bodies becomes a rigid one degrees-of-freedom. Relative displacement condition, in vector form is

$$\overline{O_j O_i} = \overline{0}, \qquad (13)$$

and conditions of invariable relative orientation are:

$$\overline{i_i} \cdot \overline{i_j} = \left(\overline{i_i} \cdot \overline{i_j}\right)_0, \; \overline{j_i} \cdot \overline{j_j} = \left(\overline{j_i} \cdot \overline{j_j}\right)_0, \; \overline{k_i} \cdot \overline{k_j} = \left(\overline{k_i} \cdot \overline{k_j}\right)_0 \qquad (14)$$

Index "0" from right part of above relations corresponds to initial moment and it shows that inner products from left side are constants. If relation (5) is written in the form

$$\begin{Bmatrix} \overline{i_i} \\ \overline{j_i} \\ \overline{k_i} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{i,j} \begin{Bmatrix} \overline{i_j} \\ \overline{j_j} \\ \overline{k_j} \end{Bmatrix} \qquad (15)$$

relations (14) become:

$$a_{11}^{ij} = \left(a_{11}^{ij}\right)_0, \; a_{22}^{ij} = \left(a_{22}^{ij}\right)_0, \; a_{33}^{ij} = \left(a_{33}^{ij}\right)_0 \qquad (16)$$

2.2.3 Spherical joint

Spherical joint reduces the number of degrees-freedom with three units. Vector form of constraint is condition (13).

2.2.4. Linkage of translation

When the coupling mechanism allows translations in three some directions the number of degrees-freedom is reduced with three units and constraints are of the (16) form.

2.2.5. Connection with flexible cable

Coupling mechanism with flexible cable reduces the number of degrees-freedom with one unit. The distance between points of connection of flexible cable is a constant one, and conditions is

$$\left|\overline{O_j O_i}\right| = \left|\overline{O_j O_i}\right|_0 \qquad (17)$$

2.3. Constraints expressed by velocities

When constraints are expressed by velocities (velocities of translations or angular velocities) the motion is described with Lagrange equations for nonholonomic systems. Coupling mechanism can be analyzed from the point of view of allowed mobility. Bellow translations mobility and rotations mobility are analyzed.

2.3.1. Translation conditions

If the coupling mechanism allows translations in three some directions, the number of constraints which correspond to translations is zero. If the coupling mechanism allows translations in two directions of vectors $\overline{t_{j1}}\left(t_{j1x}, t_{j1y}, t_{j1z}\right)$ and $\overline{t_{j2}}\left(t_{j2x}, t_{j2y}, t_{j2z}\right)$ with components in the system of (*j*) body, than the constraint is expressed by inner product

$$\left(\overline{v_{Oi}} - \overline{v_{Oj}}\right) \cdot \left(\overline{t_{j1}} \times \overline{t_{j2}}\right) = 0 \qquad (18)$$

The above relation can be written in matrix form with components of vectors from the trihedral of (j) body. Velocity $\overline{v_{Oi}}$ is expressed with components in reference frame of (i) body, velocity $\overline{v_{Oj}}$ is expressed with components in reference frame of (j) body and (18) becomes

$$\begin{aligned}&\{v_{Oi}\}_i-\{v_{Oj}\}_j=\left([A_{ij}]\{v_{Oi}\}_i-\{v_{Oj}\}_j\right)_j=\\&=[A_{ij}](\{v_{Ci}\}_0+[\hat{\omega}_{i,0}]\{C_iO_i\})-(\{v_{Cj}\}_0+[\hat{\omega}_{j,0}]\{C_jO_j\})=\\&[A_{ij}]([A_{i0}]\{v_{Ci}\}+[\hat{\omega}_{i,0}]\{C_iO_i\})-([A_{j0}]\{v_{Cj}\}_0+[\hat{\omega}_{j,0}]\{C_jO_j\})\end{aligned}\qquad(19)$$

In the above equations velocities $\{v_{Ci}\}$, $\{v_{Cj}\}$ of points C_i, respectively C_j are expressed with components from $O_{0x0y0z0}$ reference frame, matrices $\lfloor\hat{\omega}_{i,0}\rfloor$, $\{C_iO_i\}$ are expressed with components in reference frame of (j) body. For cross products, the vector

$$\{\omega\}=\begin{Bmatrix}\omega_x\\\omega_y\\\omega_z\end{Bmatrix}\qquad(20)$$

is associated to the antiskew matrix,

$$[\hat{\omega}]=\begin{bmatrix}0&-\omega_z&\omega_y\\\omega_z&0&-\omega_x\\-\omega_y&\omega_x&0\end{bmatrix}.\qquad(21)$$

Matrix form of (18) becomes

$$\begin{aligned}&\lfloor A_{ji}\rfloor([A_{i0}]\{v_{Ci}\}+\lfloor\hat{\omega}_{i,0}\rfloor\{C_iO_i\})-\\&-([A_{j0}]\{v_{Cj}\}+[\hat{\omega}_{j,0}]\{C_jO_j\}))]^T[[\hat{t}_{j1}]\{t_{j2}\}]=0\end{aligned}\qquad(22)$$

When the coupling mechanism allows one translation in direction of $\overline{t_j}(t_{jx},t_{jy},t_{jz})$ vector which is expressed with components in the trihedral of (j) body, two scalar relations which correspond to vector form

$$\overline{v_{Oi}}-\overline{v_{Oj}}=\lambda_t\overline{t_j}\qquad(23)$$

can be written and this shows co-linearity of relative velocity of body (i) with body (j) with $\overline{t_j}$ vector. Matrix form of (23) condition is

$$\begin{aligned}&\lfloor A_{ji}\rfloor([A_{i0}]\{v_{Ci}\}+\lfloor\hat{\omega}_{i,0}\rfloor\{C_iO_i\})-\\&-([A_{j0}]\{v_{Cj}\}+[\hat{\omega}_{j,0}]\{C_jO_j\})=\lambda_t\{t_j\}\end{aligned}\qquad(24)$$

<u>2.3.2. Rotation conditions</u>

When the coupling mechanism allows rotations with respect to some three directions, the number of constrains is zero.

If the coupling mechanism allows rotations with respect to directions determined by vectors $\overline{r_{j1}}(r_{j1x},r_{j1y},r_{j1z})$ and $\overline{r_{j2}}(r_{j2x},r_{j2y},r_{j2z})$, which are expressed with components in (j) body reference frame, condition

$$(\overline{\omega_{i,0}}-\overline{\omega_{j,0}})\cdot(\overline{r_{j1}}\times\overline{r_{j2}})=0\qquad(25)$$

can be written, or in matrix form,

$$[[A_{ji}]\{\omega_{i,0}\}-\{\omega_{j,0}\}]^T[[\hat{r}_{j1}]\{r_{j2}\}]=0\qquad(26)$$

If the coupling mechanism allows one rotation with respect to the direction determined by vector $\overline{r_j}(r_{jx},r_{jy},r_{jz})$, which is expressed with components in (j) body reference frame, than two scalar conditions which are included in vector form

$$\overline{\omega_{i,0}}-\overline{\omega_{j,0}}=\lambda_r\overline{r_j}\,,\qquad(27)$$

can be written, or in matrix form,

$$\lfloor A_{ji}\rfloor\{\omega_{i,0}\}-\{\omega_{j,0}\}=\lambda_r\{r_j\}\qquad(28)$$

3. LAGRANGE EQUATIONS FOR HOLONOMIC SYSTEMS WITH DEPENDENT VARIABLES

For a non-holonomic rheonomic system Lagrange equations for h coordinates

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial\dot{q}_k}\right)-\frac{\partial E}{\partial q_k}=Q_k+\sum_{i=1}^{p}\lambda_i a_{ik},(k=1,2,\cdots,h)\quad(29)$$

are completed with constraints

$$\sum_{k=1}^{h}a_{ik}\dot{q}_k+b_i=0,\quad(i=1,2,\cdots p)\qquad(30)$$

which can be written in the form

$$\sum_{k=1}^{h}a_{ik}\mathrm{d}q_k+b_i\mathrm{d}t=0,\quad(i=1,2,\cdots p)\qquad(31)$$

By solving the system of h equations (29) and p equations (31), q_k coordinates and λ_i multipliers are found. In the case of a holonomic system constraints have the form

$$\Phi_i(q_1,\cdots,q_h,t)=0,(i=1,2,\cdots p)\qquad(32)$$

and differential form is obtained as

$$\sum_{k=1}^{h} \frac{\partial \Phi_i}{\partial q_k} dq_k + b_i dt = 0, \quad (i = 1,2,\cdots p) \tag{33}$$

From (33) and (30) it follows

$$a_{ik} = \frac{\partial \Phi_i}{\partial q_k}, \tag{34}$$

and equations (29) become

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k + \frac{\partial}{\partial q_k}\sum_{i=1}^{p}\lambda_i \Phi_i, \quad (k = 1,2,\cdots,h) \tag{35}$$

If the function

$$U_\phi = \sum_{i=1}^{p} \lambda_i \Phi_i \tag{36}$$

is introduced, than, equations (35) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E}{\partial \dot{q}_k}\right) - \frac{\partial E}{\partial q_k} = Q_k + \frac{\partial U_\Phi}{\partial q_k}, \quad (k = 1,2,\cdots,h) \tag{37}$$

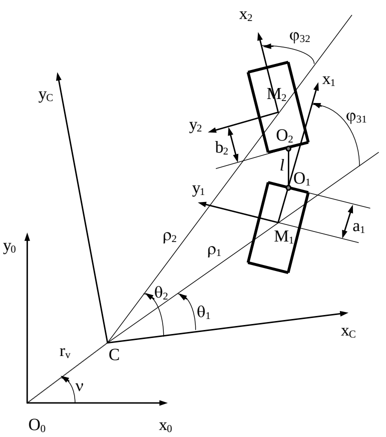

Fig. 2. System of tethered bodies

From the h above equations and p constraints (32) functions which correspond to h generalized coordinates q_k and to p multipliers λ_i are determined.

In the case of plane relative motion of two tethered bodies, scalar condition (32) is

$$\left|\overline{\mathrm{O_1O_2}}\right| = l = const. \tag{38}$$

With components of the vector (fig. 2.)

$$\overline{\mathrm{O_2O_1}} = \overline{\mathrm{CM_1}} + \overline{\mathrm{M_1O_1}} - \left(\overline{\mathrm{CM_2}} + \overline{\mathrm{M_2O_2}}\right) \tag{39}$$

on the axes of C_{xcyc} trihedral the bellow constraint

$$\begin{aligned}
\Phi \equiv\; & \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\theta_1 - \theta_2) + 2\rho_1 a_1 \cos\varphi_{31} + \\
& + 2\rho_1 b_2 \cos(\theta_1 - \theta_2 - \varphi_{32}) - 2\rho_2 a_1 \cos(\theta_1 - \theta_2 + \varphi_{31}) - \\
& - 2\rho_2 b_2 \cos\varphi_{32} + 2a_1 b_2 \cos(\theta_1 - \theta_2 + \varphi_{31} - \varphi_{32}) + \\
& + a_1^2 + b_1^2 - l^2 = 0
\end{aligned} \tag{40}$$

is obtained, which is used to form the function

$$U_\Phi = \lambda\Phi\,. \tag{41}$$

From equations (31) and the above constraints, equations of motion for the system of two tethered bodies are obtained:

$$\begin{aligned}
& m_1\left[\ddot{\rho}_1 - \rho_1\left(n + \dot{\theta}_1\right)^2\right] = -m_1 n^2 \rho_1\left(1 - 3\cos^2\theta_1\right) + \\
& + \frac{3n^2}{r_v}\left[\cos\theta_1 + \frac{\rho_1}{r_v}\left(1 - 5\cos^2\theta_1\right)\right]\cdot\left\{\frac{1}{2}(A_1 + B_1 + C_1) - \right. \\
& - \frac{3}{2}\left(A_1 c_{31}^2 + B_1 s_{31}^2\right) + \frac{3}{2}(A_1 - B_1)\frac{\rho_1}{r_v}\left[\frac{\rho_1}{r_v}\left(c_{31}^2 - s_{31}^2\right)\sin^2\theta_1 - \right. \\
& \left.\left. - \left(1 - \frac{\rho_1}{r_v}\cos\theta_1\right)s_{31}c_{31}\sin\theta_1\right]\right\} + n^2\left[1 - 3\frac{\rho_1}{r_v}\cos\theta_1 - \frac{3}{2}\frac{\rho_1^2}{r_v^2}\left(1 - 5\cos^2\theta_1\right)\right]\cdot \\
& \cdot\frac{3(A_1 - B_1)}{r_v}\left[\frac{\rho_1}{r_v}\left(c_{31}^2 - s_{31}^2\right)\sin^2\theta_1 - \frac{1}{2}\left(1 - 2\rho_1\cos\theta_1\right)s_{31}c_{31}\sin\theta_1\right] + \\
& + 2\lambda\left[\rho_1 - \rho_2\cos(\theta_1 - \theta_2) + a_1\cos\varphi_{31} + b_2\cos(\theta_1 - \theta_2 - \varphi_{32})\right] + Q_{\rho_1}^*\,;
\end{aligned}$$

$$\begin{aligned}
& m_2\left[\ddot{\rho}_2 - \rho_2\left(n + \dot{\theta}_2\right)^2\right] = -m_2 n^2 \rho_2\left(1 - 3\cos^2\theta_2\right) + \\
& + \frac{3n^2}{r_v}\left[\cos\theta_2 + \frac{\rho_2}{r_v}\left(1 - 5\cos^2\theta_2\right)\right]\cdot\left\{\frac{1}{2}(A_2 + B_2 + C_2) - \right. \\
& - \frac{3}{2}\left(A_2 c_{32}^2 + B_2 s_{32}^2\right) + \frac{3}{2}(A_2 - B_2)\frac{\rho_2}{r_v}\left[\frac{\rho_2}{r_v}\left(c_{32}^2 - s_{32}^2\right)\sin^2\theta_2 - \right. \\
& \left.\left. - \left(1 - \frac{\rho_2}{r_v}\cos\theta_2\right)s_{32}c_{32}\sin\theta_2\right]\right\} + n^2\left[1 - 3\frac{\rho_2}{r_v}\cos\theta_2 - \frac{3}{2}\frac{\rho_2^2}{r_v^2}\left(1 - 5\cos^2\theta_2\right)\right]\cdot \\
& \cdot\frac{3(A_2 - B_2)}{r_v}\left[\frac{\rho_2}{r_v}\left(c_{32}^2 - s_{32}^2\right)\sin^2\theta_2 - \frac{1}{2}\left(1 - 2\rho_2\cos\theta_2\right)s_{32}c_{32}\sin\theta_2\right] + \\
& + 2\lambda\left[\rho_2 - \rho_1\cos(\theta_1 - \theta_2) - b_2\cos\varphi_{32} - a_1\cos(\theta_1 - \theta_2 + \varphi_{31})\right] + Q_{\rho_2}^*\,;
\end{aligned}$$

$$\begin{aligned}
& m_1\left[\rho_1^2\ddot{\theta}_1 + 2\rho_1\dot{\rho}_1\left(n + \dot{\theta}_1\right)\right] = -\frac{3}{2}m_1 n^2\rho_1^2\sin 2\theta_1 + \frac{3}{2}n^2(A_1 - B_1)\frac{\rho_1^2}{r_v^2}\cdot \\
& \cdot\left(3\sin\theta_1 - \frac{5}{2}\frac{\rho_1}{r_v}\sin 2\theta_1\right)\left[\frac{\rho_1}{r_v}\left(c_{31}^2 - s_{31}^2\right)\sin 2\theta_1 - \left(\cos\theta_1 - \frac{\rho_1}{r_v}\cos 2\theta_1\right)s_{31}c_{31}\right] \\
& + 2\lambda\left[\rho_1\rho_2\sin(\theta_1 - \theta_2) - \rho_2 b_2\sin(\theta_1 - \theta_2 - \varphi_{32}) + \rho_2 a_1\sin(\theta_1 - \theta_2 + \varphi_{31}) - \right. \\
& \left. - a_1 b_2\sin(\theta_1 - \theta_2 + \varphi_{31} - \varphi_{32})\right] + Q_{\theta_1}^*\,;
\end{aligned} \tag{42}$$

$$\begin{aligned}
& m_2\left[\rho_2^2\ddot{\theta}_2 + 2\rho_2\dot{\rho}_2\left(n + \dot{\theta}_2\right)\right] = -\frac{3}{2}m_2 n^2\rho_2^2\sin 2\theta_2 + \frac{3}{2}n^2(A_2 - B_2)\frac{\rho_2^2}{r_v^2}\cdot \\
& \cdot\left(3\sin\theta_2 - \frac{5}{2}\frac{\rho_2}{r_v}\sin 2\theta_2\right)\left[\frac{\rho_2}{r_v}\left(c_{32}^2 - s_{32}^2\right)\sin 2\theta_2 - \left(\cos\theta_2 - \frac{\rho_2}{r_v}\cos 2\theta_2\right)s_{32}c_{32}\right] + \\
& + 2\lambda\left[-\rho_1\rho_2\sin(\theta_1 - \theta_2) + \rho_1 b_2\sin(\theta_1 - \theta_2 - \varphi_{32}) - \rho_2 a_1\sin(\theta_1 - \theta_2 + \varphi_{31}) + \right. \\
& \left. + a_1 b_2\sin(\theta_1 - \theta_2 + \varphi_{31} - \varphi_{32})\right] + Q_{\theta_2}^*\,;
\end{aligned}$$

$$\begin{aligned}
& C\ddot{\varphi}_{31} = \frac{3}{2}(A_1 - B_1)n^2\left[1 - 3\frac{\rho_1}{r_v}\cos\theta_1 - \frac{3}{2}\frac{\rho_1^2}{r_v^2}\left(1 - 5\cos^2\theta_1\right)\right]\cdot \\
& \cdot\left[\left(1 - 2\frac{\rho_1^2}{r_v^2}\sin^2\theta_1\right)\sin 2\varphi_{31} - \frac{\rho_1}{r_v}\left(1 - \frac{\rho_1}{r_v}\cos\theta_1\right)\sin\theta_1\cos 2\varphi_{31}\right] + \\
& + 2\lambda\left[-\rho_1 a_1\sin\varphi_{31} + \rho_2 a_1\sin(\theta_1 - \theta_2 + \varphi_{31}) - \right. \\
& \left. - a_1 b_2\sin(\theta_1 - \theta_2 + \varphi_{31} - \varphi_{32})\right] + Q_{\varphi_{31}}^*\,;
\end{aligned}$$

$$\begin{aligned}
& C\ddot{\varphi}_{32} = \frac{3}{2}(A_2 - B_2)n^2\left[1 - 3\frac{\rho_2}{r_v}\cos\theta_2 - \frac{3}{2}\frac{\rho_2^2}{r_v^2}\left(1 - 5\cos^2\theta_2\right)\right]\cdot \\
& \cdot\left[\left(1 - 2\frac{\rho_2^2}{r_v^2}\sin^2\theta_2\right)\sin 2\varphi_{32} - \frac{\rho_2}{r_v}\left(1 - \frac{\rho_2}{r_v}\cos\theta_2\right)\sin\theta_2\cos 2\varphi_{32}\right] + \\
& + 2\lambda\left[\rho_2 b_2\sin\varphi_{32} + \rho_1 b_2\sin(\theta_1 - \theta_2 - \varphi_{32}) + \right. \\
& \left. + a_1 b_2\sin(\theta_1 - \theta_2 + \varphi_{31} - \varphi_{32})\right] + Q_{\varphi_{32}}^*\,.
\end{aligned}$$

Equations (42) are valid only if the tension in the flexible cable is positive one. When the cable isn't stretched equation of motion are obtained from (42) for zero value of λ parameter. Tension in the cable can be computed using one equation for one isolated body. In figures 3-5 results obtained by integration of (42) equations are presented ([7]).
Two cylindrical bodies with radii of bases of 10m, respectively 20m, connected by one cable of 500m length are considered. Relative motion of two bodies is studied with respect to a circular motion of 10500km radius.

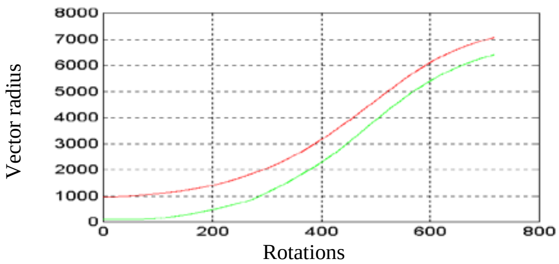

Fig. 3

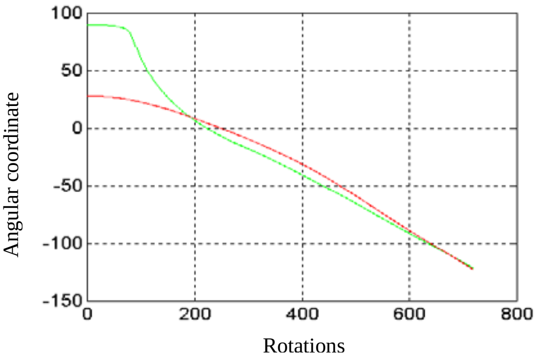

Fig. 4

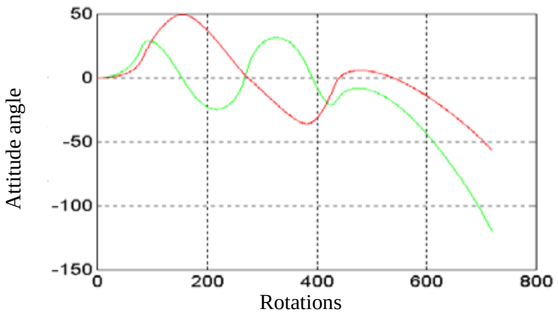

Fig. 5

4. CONCLUSIONS

The problem of kinematics for systems of bodies are solved using analyzes of coupling mechanism under the aspect of number of degrees-of-freedom. The motion in central gravitational field is studied with respect a movable reference frame with origin on a circular orbit. The problem of dynamics of bodies system is solved using Lagrange equations of motion with multipliers and constraints. For the system of two tethered bodies obtained results by integration of motion equations are presented. Models and elaborated method allow solving of a great number of problems of bodies' systems dynamics in gravitational field.

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