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COMPLEX MODULUS AND ENERGY DISSIPATION IN DAMPED SANDWICH STRUCTURES

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Abstract: *The equivalence of the standard method of damped normal modes and the method of modal strain energies in computation of the response of rather heavily damped sandwich structures is demonstrated. Estimation of the response of damped sandwich beams to harmonic excitation is here based on the truncated integral modal transformation in which the base functions are the modal data from the associated undamped system. This approach is in fact a reinterpretation of the concept of damped normal modes so that it allows to carry out all the necessary numerical computations only in the real domain while previous formulation required numerical treatment in the complex domain.*

Key words: sandwich beams, complex modulus, damped normal modes, strain energy ratio

1. INTRODUCTION

Material damping significantly influences the response of vibrating structures to external excitation. Despite the steady progress in experimental methods the modelling of damping of structures persists to remain as the most problematic issue connected with modelling and simulating of the response of complex structures. One of the most popular and widespread methods of modelling the damping of real structures vibrating under harmonic forces is the use of complex moduli. The vibroelastic damping behaviour of applied materials is incorporated in equations of motion through the constitutive law, in which for the real modulus E the complex modulus E_R+iE_I is susbstituded. The most simple model of this type with constant moduli E_R and E_I is referred to as the structural damping, the hysteretic damping or as the frequency-independent damping. Thus, when the problem is formulated for vibrating continuum, the stiffness operator becomes complex and consequently also the resulting response is given by complex valued vector function. It is necessary to add, that this model is admitted only under harmonic regime [1]. For the formal justification of the use of the concept of complex damping the correspondence principle is often quoted in western literature. In former Soviet sources the Sorokin`s hypothesis is frequently mentioned in the same context.

The use of complex moduli in structural analysis is evidently rather a rough simplification. Engineers are highly satisfied being able to estimate only the global manifestation of the dissipative processes in the response. However, this simplification calls forth at least the following conceptual difficulties:

First, the eigenvalues of the appropriate eigenvalue problem always appear in pairs - α_r +iω_r and α_r +iω_r, so that half of the solutions exhibit asymptotic instability of flutter type. This mathematical inconsistency is usually obviated by admitting only the harmonic regime or more or less equivalently by saying that from physical reasons the asymptotic stability is required besides of the fulfilment of the initial and boundary conditions. Then the unstable “half” of solutions is excluded by the choice of zero valued integration constants.

Second, hysteretic damping used in engineering applications often results in physiccally impossible non-causal response, namely when transient excitation is considered. The acceptability of some degree of non-causality for reasonable damping parameters has been discussed by Lundén [2] with the conclusion that it is the price for simplifying the modelling of damping and it`s numerical treatment. Needless to admit, that in conventional analysis [3-5] this is only one of many of other simplifications about the linearity, isotropy, boundary conditions, etc.

2. GOVERNING EQUATIONS OF DAMPED SANDWICH BEAM

Passive damping of vibrating structures can be achieved by using the sandwich type of structures, consisting of a viscoelastic core material constrained between two elastic face layers. Basic idea behind the constrained layer damping technique is the dissipation of energy through the shear deformation induced in the viscoelastic core due to flexure. The aim of the designer is to allow as much shear strain in the core as possible to obtain sufficient level of dissipation. Classical theory of flexural vibration of sandwich beams was elaborated first by DiTaranto [6]. Mead and Markuš [7] in series of works have contributed substantially to the theory by putting it to a form, which is nowadays considered to be a standard. According to this theory only harmonic regime is considered, so that in the sense explained above it is legitimate to introduce the hysteretic damping of the core through the complex shear modulus

$$G_2^* = G_2(1 + i\,\beta)\,, \qquad (1)$$

where the β is the core loss factor, which is believed to be a material constant. For simplicity, we assume here that G_2 and β are frequency-independent constants. Damped harmonic vibration of one-dimensional continuum is then described by a model of the response in standard form

$$\mathbf{L}_T\mathbf{W} = \mathbf{f}\,, \quad \mathbf{L}_T = \mathbf{K} - p^2\mathbf{M} + i\,\mathbf{B}\,, \quad (2)$$

where **K**, **M**, **B** are the matrices of linear differential operators corresponding to dynamic stiffness, mass and damping, **W** and **f** are vector functions of the response amplitude and of external excitation, respectively. The function **W** belongs to the space of comparison functions, satisfying all the boundary conditions, **f** has only real components and the p is non-dimensional frequency of harmonic excitation.

Mass operator **M** is diagonal. Assuming, that the dissipation in the face layers is negligible, the damping operator for model (1) becomes

$$i\,\mathbf{B} = i\,\beta\,\mathbf{K}_G\,, \qquad (3)$$

where the $\mathbf{K}_G$ is essentially an extraction of that part of the stiffness operator **K**, which is proportional to the core shear modulus G_2.

The actual form of operators for the case of sandwich beam has the structure

$$\mathbf{K} = \begin{bmatrix} D^4 - ga^2D^2 & -gaD & gaD \\ gaD & g - t_1D^2 & -g \\ -gaD & -g & g - t_3D^2 \end{bmatrix}$$

$$\mathbf{K}_G = \begin{bmatrix} -ga^2D^2 & -gaD & gaD \\ gaD & g & -g \\ -gaD & -g & g \end{bmatrix},$$

where a, t_1, t_3 are parameters characterising the geometrical configuration of the sandwich beam, while the shear modulus G_2 appears as factor in the non-dimensional parameter g, reflecting the stiffness of both the core and the face layers. Explicitly, the damping matrix is of form

$$i\,\mathbf{B} = i\,\beta\mathbf{K}_G = i\,\beta\,g \begin{bmatrix} -a^2D^2 & -aD & aD \\ aD & 1 & -1 \\ -aD & -1 & 1 \end{bmatrix}.$$

Note the nice regular structure of matrices, which form the operators **K**, $\mathbf{K_G}$ and indirectly also the damping operator i**B**. Each of these matrices is symmetric in even order differential expressions and antisymmetric in expressions of odd order. However, when combined into operator $\mathbf{L}_T$ defined in (2), due to presence of the factor $(1+i\beta)$ over the diagonal and at the same time also under the diagonal, the symmetries are completely lost for $\beta \neq 0$ and the operator $\mathbf{L}_T$ is non-Hermitian.

2. EXACT RESPONSE OF DAMPED SANDWICH BEAM

The scalar product in the space of complex valued vector functions is defined as

$$(\mathbf{X},\mathbf{Y}) = \int_0^1 \mathbf{X}^T(\eta)\cdot\overline{\mathbf{Y}}(\eta)\,d\eta, \qquad (4)$$

where ^{T} denotes transposition and the overbar denotes the complex conjugate. The adjoint operator corresponding to the operator

$$\mathbf{L}_T = \mathbf{K} + i\,\mathbf{B} - \lambda\,\mathbf{M} \qquad (5a)$$

reads

$$\mathbf{L}^*_T = \mathbf{K} - i\mathbf{B} - \overline{\lambda}\,\mathbf{M}, \qquad (5b)$$

as it is easily verified using the Green`s identity

$$(\mathbf{L}_T\mathbf{U},\mathbf{V}) = (\mathbf{U},\mathbf{L}^*_T\mathbf{V}). \qquad (6)$$

At this point the $\mathbf{L}_T^*$ is the formal adjoint of $\mathbf{L}_T$, as we have not defined the precise domains of operators $\mathbf{L}_T$ and $\mathbf{L}_T^*$. The variationaly optimal domains of operators $\mathbf{L}_T$ and $\mathbf{L}_T^*$ require fulfilment of fundamental boundary conditions for the "left" comparison functions **U** as well as consideration of adjoint boundary conditions for the "right" comparison functions **V**. The data (eigenvalues and eigenfunctions) from both the fundamental and adjoint eigenvalue problems

$$\begin{aligned}(\mathbf{K} + i\,\mathbf{B} - \lambda\,\mathbf{M})\cdot\mathbf{U} &= \mathbf{0}\\ (\mathbf{K} + i\,\mathbf{B} - \overline{\lambda}\,\mathbf{M})\cdot\mathbf{U}^* &= \mathbf{0}\end{aligned} \qquad (7)$$

are necessary to obtain the exact response

$$\mathbf{W}(\eta) = \sum_{k=1}^{\infty}\frac{(\mathbf{f},\mathbf{M}\mathbf{U}^*_k)}{\lambda-\lambda_k}\,\mathbf{U}_k\,. \qquad (8)$$

The data can be scaled to fulfil the biorthogonality relations

$$\begin{aligned}(\mathbf{U}^*_j,\mathbf{M}\mathbf{U}_k) &= \delta_{jk},\\ (\mathbf{U}^*_j,[\mathbf{K} + i\,\mathbf{B}]\mathbf{U}_k) &= \delta_{jk}\lambda_k,\end{aligned} \qquad (9)$$

where δ_{jk} is the Kronecker delta.

Though **K**, **B**, **M** are formaly selfadjoint, the operator $\mathbf{L}_T$ is non-selfadjoint due to the presence of the non-Hermitian damping operator i**B**. Therefore the eigenvalues of $\mathbf{L}_T$ and $\mathbf{L}_T^*$ are in general complex-valued and there does not exist simple relation between the eigenfunctions $\mathbf{U}_k$ and adjoint eigenfunctions $\mathbf{U}_j^*$ except for the biorthogonality relation (9).

Unless the operator **B** does not commutate with the operator ($\mathbf{K}-\lambda\mathbf{M}$), both the fundamental and the adjoint eigenvalue problems (7) are to be solved numerically to obtain the exact response. To emphasize, that the exact solution is rather cumbersome and demanding, it is necessary to add, that all the computations are to be performed in the complex domain.

3. METHOD OF DAMPED NORMAL MODES

To avoid the complicated exact solution, many other alternative schemes of approximate solutions of the eigenvalue problems of the damped structures have been proposed.

The widespread method of "damped normal modes" was developed by Mead and Markuš [7]. This method claims the existence of special loading distribution for each possible beam resonance, which excites corresponding uncoupled modes and it is claimed, that at resonant frequency the beam behaves like a single-degree-of-freedom system. Instead of solving the adjoint boundary value problem, particular form of forced vibration is searched for the beam being excited by a "damped normal loading" in order to enable uncoupled modal analysis. The reasoning, given in [7], is based among others on orthogonality of damped modes under evidently inconsistent scalar product

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \int_0^1 \mathbf{X}^T(\eta) \cdot \mathbf{Y}(\eta)\, d\eta,$$

in which the complex conjugation of the generally complex function **Y** is ignored. Due to lack of firm mathematical foundation, the concept of damped normal modes can not be considered as an equivalent alternative to the exact solution (8).

Present paper is an attempt to find out the consequencies of speculations with incorrect scalar product. In other words, the question posed is, which alternate approximate solution is most close to the popular concept of damped normal modes.

An alternate method of approximate solution is developed in the following paragraph, in which the simplifications are openly declared. To our surprise, the numerical results obtained by both the damped normal modes and by the modal strain energy ratio are perfectly identical. The conclusion then is, that both methods are legitimate only as an approximation.

4. APPROXIMATE SOLUTION

It is not an easy task to find the complex fundamental and adjoint eigenfunctions $\mathbf{U_k}$ and $\mathbf{U}_j^*$, that are necessary for computing the exact solution (8) of the damped problem (2). In practical applications the eigenfunctions of the associated undamped eigenvalue boundary problem are often used to estimate the exact solution (8) as well as to estimate the dissipative behaviour of the damped sandwich beam. Let us suppose that the boundary conditions are specified so that the associated undamped problem

$$(\mathbf{K} - \lambda \mathbf{M}) \cdot \mathbf{W} = \mathbf{f},\, \ell_B[\mathbf{W}(0), \mathbf{W}(1)] = 0 \quad (10)$$

is selfadjoint and positive definite. Then the eigenvalues λ_i, $i=1,2,...$ are positive and the eigenfuctions $\mathbf{W}_i$ are real and

$$(\mathbf{W}_i, \mathbf{M}\mathbf{W}_j) = \delta_{ij}, (\mathbf{W}_i, \mathbf{K}\mathbf{W}_j) = \delta_{ij}\lambda_i . \quad (11)$$

The following analysis is based on the truncated set of first n eigenfunctions giving rise to the truncated eigenmatrix

$$\mathbf{V} = [\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_n]. \quad (12)$$

Integral modal transformation in the truncated base **V** assigns to the operators **K**, **M**, **B** and $\mathbf{L_T}$ the spectral matrix

$$\mathbf{S} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n], \quad (13)$$

the unity matrix **I** and the matrices **H** and **Z** by relations

$$\begin{aligned} (\mathbf{V}, \mathbf{KV}) &= \mathbf{S}, \\ (\mathbf{V}, \mathbf{MV}) &= \mathbf{I}, \\ (\mathbf{V}, \mathbf{BV}) &= \mathbf{H}, \\ \mathbf{Z} &= \mathbf{S} - p^2\mathbf{I} + i\,\mathbf{H}, \end{aligned} \quad (14)$$

where the **(.,.)** is the scalar product (4). Matrices **H** and **Z** are generally nondiagonal except for the special case when the operator **B** commutes with both the **K** and **M**. The consequence is that the discretized equations of motion are coupled, possible approximation to simplify the solution is to neglect the off-diagonal elements in matrix **H**. With approximation $h_{ij}=0$, $i \neq j$, the matrix **Z** is now diagonal:

$$\mathbf{Z} = \mathbf{S} - p^2\mathbf{I} + i\beta\mathbf{SQ}. \quad (15)$$

According to Hasselman [8] or Warburton [9] such an approximation is admissible in many problems. The diagonal elements of matrix **H** can be expressed as

$$h_{ii} = (\mathbf{W}_i, \mathbf{B}\mathbf{W}_i). \quad (16)$$

Integrating by parts we find

$$\begin{aligned} h_{ii} &= (\mathbf{W}_i, \mathbf{B}\mathbf{W}_i) = \\ &= \beta\, g \int_0^1 (aDW_i + U_i - U_3)^2 d\eta - \\ &- \beta\, g\, a[W_1(a\,DW_i + U_1 - U_3)]_0^1. \end{aligned} \quad (17)$$

Obviously the expressions h_{ii} can be interpreted as the strain energy due to the shear deformations of the core multiplied by the material loss factor β. Let us extend by unity the left side of (17) as follows:

$$h_{ii} = \frac{(\mathbf{W}_i, \mathbf{K}\mathbf{W}_i)}{(\mathbf{W}_i, \mathbf{K}\mathbf{W}_i)} (\mathbf{W}_i, \mathbf{B}\mathbf{W}_i) \quad (18)$$

and denote

$$q_{si} = \frac{P_{si}}{P_i} = \frac{(\mathbf{W}_i, \mathbf{K}_G \mathbf{W}_i)}{(\mathbf{W}_i, \mathbf{K}\mathbf{W}_i)}, \qquad (19)$$

which is the portion of the potential energy P_{si} due to shear deformation of the core and of the total potential energy P_i of the undamped sandwich beam vibrating in *i*-th mode, i.e.

$$\begin{aligned} P_{si} &= (\mathbf{W}_i, \mathbf{K}_G \mathbf{W}_i) \\ P_i &= (\mathbf{W}_i, \mathbf{K}\mathbf{W}_i). \end{aligned} \qquad (20)$$

Rearranging (18) and using (19),(20) together with the first of orthogonality relations (14) we arrive to key relation of the present analysis

$$h_{ii} = \beta\, q_{si}\, p_i^2. \qquad (21)$$

Neglecting the off diagonal terms h_{ij}, i.e. with the approximation $h_{ij}=0,\ i\neq j$ the matrix **Z** becomes diagonal

$$\mathbf{Z} = \mathbf{S} - p^2\mathbf{I} + i\beta\,\mathbf{S}\mathbf{Q}, \qquad (21)$$

where

$$\mathbf{Q} = \mathrm{diag}[q_{s1}, q_{s2}, \ldots, q_{sn}]. \qquad (22)$$

The Galerkin procedure leads to the solution

$$\mathbf{W}(\eta, t) = \sum_{k=1}^{n} \mathbf{W}_k(\eta)\, a_k(t) \qquad (23)$$

in which the generalized coordinates $a_k(t)$ are given by the uncoupled differential equations

$$\frac{d^2}{dt^2} a_k(t) + p_k^2 (1 + \mathrm{i}\beta q_{sk}) a_k(t) = = \mathbf{f}_{a,k}\, \mathrm{e}^{\mathrm{i}pt} \qquad (24)$$

where $\mathbf{f}_a = (\mathbf{V}, \mathbf{f})$. The steady state solution of (24) is

$$\mathbf{W}(\eta, t) = \sum_{k=1}^{n} \frac{(\mathbf{W}_k, \mathbf{f}_a)\mathbf{W}_k}{p_k^2 - p^2 + \mathrm{i}\beta q_{sk} p_k^2} \mathrm{e}^{ipt} \qquad (25)$$

In this way the diagonalization of the damping matrix **H**, by omitting the off-diagonal elements, approximates the non-proportionally damped beam by a special type of proportional damping. Thus, for each mode different values of proportionality constants are valid. To each mode it is possible to assign an individual damping factor, which can be percieved from the global point of view as frequency dependent quantity. With the traditional proportional damping it is not possible to individualize the measure of damping capacity from mode to mode.

5. COMPARISON OF APPROXIMATE METHODS

Mead and Markuš [7] analyzed the vibration of damped sandwich beams using a special class of forced, uncoupled and complex modes of vibration. To find the response of the damped sandwich beam, the computations according to their method are to be carried out in the complex domain. The modal loss factor is then found from complex eigenvalues by the classical relation

$$\eta_k = \frac{\mathrm{Im}(\lambda_k)}{\mathrm{Re}(\lambda_k)}. \qquad (26)$$

However, the detailed analysis of their paper [7] and the comparison of numerical results based on both the concept of damped normal modes and the approach adopted in the previous paragraph shows, that the result are completely identical. The fundamental difference is that the modal strain energy approach presented here requires computations only in the real domain, which are obviously significantly easier to perform than the numerical computations in complex domain. The modal loss factor (26) is according to (21) given by simple relation

$$\eta_k = \beta q_{sk} = \beta \frac{(\mathbf{W}_i, \mathbf{K}_G \mathbf{W}_i)}{(\mathbf{W}_i, \mathbf{K}\mathbf{W}_i)}, \qquad (27)$$

where the quantity q_{sk}, the ratio of shear deformation energy to the total deformation energy, is computed from (19), i.e. from the data of the associated undamped system. Variation of q_{sk} with the shear parameter g follows exactly the pattern of the loss factor versus the shear parameter curves known from the numerous literature on sandwich beams. The modal loss factor is then obtained from the values q_{sk} by simple multiplication by the material loss factor without the need of recalculation for new values of the material loss factor, when the task of the designer was to search for optimal core material.

6. CONCLUSION

The modal strain energy approach presented in previous paragraph 4 is comprehended as reinterpretation of the concept of damped normal modes developed by Mead and Markuš [7]. The same results are now obtained by calculations carried out entirely in the real domain, what is considerable simplification of the numerical treatment of the problems of the vibration of damped structures.

As a chalenge for future research remains the realisation of the exact computations in line with the exact solution (8), which has surprisingly never been undertaken. The possible explanation is in the fact, that the exact solution requires except for the fundamental eigenfunctions also the adjoint eigenfunctions, resulting from yet undeveloped adjoint boundary conditions. In any case, there is probably no real chance to avoid the necessity of solving both the fundamental and the adjoint eigenvalue problems (7) in attempt to find the exact solution (8) or to carry out decoupled modal analysis. Any trics and simplifications to avoid the demanding effort to obtain the exact solution (8) lead to compromises, which should be responsibly and critically evaluated.

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