4th International DAAAM Conference
"INDUSTRIAL ENGINEERING – INNOVATION AS COMPETITIVE EDGE FOR SME"
29 - 30th April 2004, Tallinn, Estonia

LINEAR STABILITY ANALYSIS OF SHEAR FLEXIBLE THIN-WALLED BEAMS

Lanc, D.; Turkalj, G. & Brnić, J.

Abstract: *This paper presents finite element based numerical stability analysis of thin-walled beam structures. Using the linearized virtual work principle with assumption of large displacements, large rotation but small strains the finite element equation is derived. Effects of cross sectional shear deformations are also taken into account. To include large rotation effects the non-linear displacement field of cross-section is used. A new two-node shear flexible finite element with seven degrees of freedom per node is developed. Complete exact 14×14 elastic and geometric stiffness matrices are evaluated. A very own computer program THINWALL SHEAR is developed. Obtained solutions are for verification compared with analytical and numerical results of the other authors available from literature.*

Key words: *large displacements, large rotations, shear deformations, thin-walled beam*

1. INTRODUCTION

Tendency to an optimal construction and to reduction of product cost appeals for using thin-walled structure members because they offer a high performance for a minimum weight. Complexity of their behaviour especially from the view of stability loosing, imposes numerical modelling as adequate method because theoretical solutions are limited on cases of simple geometry.
Linear stability analysis treats stability problem as an eigenvalue one and such approach try to determine the instability load in a direct manner without calculating the deformations. A critical buckling load belongs to the lowest eigenvalue and corresponding eigenvector represents a critical or buckling mode. Such analysis supposes an ideal structure and loading condition what in other words means that eminence of deformations before reaching buckling load are not possible. Prediction of critical buckling load is usually also prediction of limit load carrying capacity.
Stability analysis concerning large spatial rotation is very complex problem because of non vectoral nature of large rotations. Using standard linear displacement field torsional moment is of semitangental and bending moments are of quasitangental character so they induce non compatible moments during large spatial rotation. In this work non-linear displacement field is used which include large rotation effects. Derived geometric stiffness matrix of thin-walled beam finite element supposes all internal moments of semitangental character.
This work also proposes that beam member is prismatic and straight, material is isotropic, cross section is non-deformable in his own plane but it is possible to warp, external loads are conservative and constitutive equations are linear.

2. BASIC CONSIDERATIONS

2.1 Nonlinear displacement field

Cross section displacements consist of seven components, three translational w_o, u_s, v_s, three rotational components $\varphi_x, \varphi_y, \varphi_z$ and also seventh component θ of cross sectional warping.
In used right handed Cartesian coordinate system (*z, x, y*), axis *z* coincidents with beam axis passing through the centroids O of cross sections. Coordinate axes *x* and *y* are the principal axes of inertia of cross section.
Total displacement field is:

$$\mathbf{U}_{uk}^{\mathrm{T}} = \left\{ w + \tilde{w} \quad u + \tilde{u} \quad v + \tilde{v} \right\}, \tag{1}$$

where *w*, *u* and *v* are linear displacement field components:

$$\begin{aligned} w &= w_o - y\,\varphi_x - x\,\varphi_y - \omega\theta, \\ u &= u_s - \left(y - y_s\right)\varphi_z, \\ v &= v_s + \left(x - x_s\right)\varphi_z \end{aligned} \tag{2}$$

and $\tilde{w}$, $\tilde{u}$ and $\tilde{v}$, are second order components:

$$\begin{aligned} \tilde{w} &= \frac{1}{2}\left[\varphi_z\varphi_x\left(x - x_s\right) + \varphi_z\varphi_y\left(y - y_s\right)\right], \\ \tilde{u} &= \frac{1}{2}\left[\varphi_z^2 x_s + \varphi_x\varphi_y y - \left(\varphi_z^2 + \varphi_y^2\right)x\right], \\ \tilde{v} &= \frac{1}{2}\left[\varphi_z^2 y_s + \varphi_x\varphi_y x - \left(\varphi_z^2 + \varphi_x^2\right)y\right] \end{aligned} \tag{3}$$

Using non linear displacement field corresponding Green-Lagrange strain tensor is:

$$\varepsilon_{ij} \cong e_{ij} + \eta_{ij} + \tilde{e}_{ij} \tag{4}$$

with components defined as (Chang et.al., 1996.):

$$\begin{aligned} e_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}), \\ \eta_{ij} &= \frac{1}{2}\left(u_{k,i}\,u_{k,j}\right), \\ \tilde{e}_{ij} &= \frac{1}{2}\left(\tilde{u}_{i,j} + \tilde{u}_{j,i}\right) \end{aligned} \tag{5}$$

Cross sectional stress resultants generally consist of: axial force F_z, shear forces F_x, F_y, torsional moment M_z, bending moments M_x, M_y and bimoment M_ω:

$$F_z = \int_A \sigma_z\, dA,$$

$$F_x = \int_A \tau_{zx}\, dA,$$

$$F_y = \int_A \tau_{zy}\, dA,$$

$$M_z = \int_A \left[\tau_{zy}(x - x_s) - \tau_{zx}(y - y_s) \right] dA,$$

$$M_x = \int_A \sigma_z y \, dA,$$

$$M_y = -\int_A \sigma_z x \, dA,$$

$$M_\omega = \int_A \sigma_z \omega \, dA.$$

The torsional moment is sum of St.Venant or uniform torsional moment T_{SV} and warping or nonuniform torsional moment T_ω.

Due to restricted cross-sectional warping an additional component known as Wagner coefficient (Yang & Kuo, 1994) is occurred and it can be expressed as :

$$\bar{K} = \int_A \sigma_z \left[(x - x_s)^2 + (y - y_s)^2 \right] dA \qquad (6)$$

or:

$$\bar{K} = \alpha_z F_z + \alpha_x M_x + \alpha_y M_y + \alpha_\omega M_\omega \qquad (7)$$

Detailed expressions of coefficients $\alpha_z, \alpha_x, \alpha_y, \alpha_\omega$ can be found in reference (Kim et.al., 1994).

In the case when shear deformation due to F_x, F_y and T_ω are taken into account we have:

$$\begin{aligned} &\frac{dv_s}{dz} + \varphi_x = \bar{\gamma}_{zy}, \\ &\frac{du_s}{dz} - \varphi_y = \bar{\gamma}_{zx}, \\ &\frac{d\varphi_z}{dz} + \theta = \bar{\gamma}_\omega \end{aligned} \qquad (8)$$

In the plane *x-z*, according to Fig. 1, we have:

$$\begin{aligned} &\frac{du_s}{dz} - \beta_y = \frac{du_s}{dz} - \varphi_y = \bar{\gamma}_{xz}, \\ &M_y = EI_y \frac{d\beta_y}{dz} = EI_y \frac{d\varphi_y}{dz}, \\ &F_x = \bar{\tau}_{zx} \cdot A_x = G \cdot \bar{\gamma}_{xz} \cdot A_x = \frac{GA}{k_x}\left(\frac{du_s}{dz} - \varphi_y \right) \end{aligned} \qquad (9)$$

and analogously follows for bending in *z-y* plane:

$$\begin{aligned} &\frac{dv_s}{dz} - \beta_x = \frac{dv_s}{dz} + \varphi_x = \bar{\gamma}_{zy}, \\ &M_x = -EI_x \frac{d\beta_x}{dz} = EI_x \frac{d\varphi_x}{dz}, \\ &F_y = \bar{\tau}_{zy} \cdot A_y = G \cdot \bar{\gamma}_{zy} \cdot A_y = \frac{GA}{k_y}\left(\varphi_x + \frac{dv_s}{dz} \right) \end{aligned} \qquad (10)$$

and for torsion:

$$\begin{aligned} &\frac{d\varphi_z}{dz} + \theta = \bar{\gamma}_\omega, \\ &M_\omega = EI_\omega \frac{d\theta}{dz}; \quad T_{SV} = GJ \frac{d\varphi_z}{dz}, \\ &T_\omega = G\bar{\gamma}_\omega J_\omega = \frac{GJ}{k_\omega}\left(\frac{d\varphi_z}{dz} + \theta \right) \end{aligned} \qquad (11)$$

In relations above $\bar{\gamma}_{xz}, \bar{\gamma}_{zy}, \bar{\gamma}_\omega$ are average values of shear deformations; $\bar{\tau}_{xz}, \bar{\tau}_{zy}$ are average values of shear stresses; A_x, A_y, J_ω are shear areas with respect to *x*, *y*, *ω* and k_x, k_y, k_ω are flexible shear coefficients.

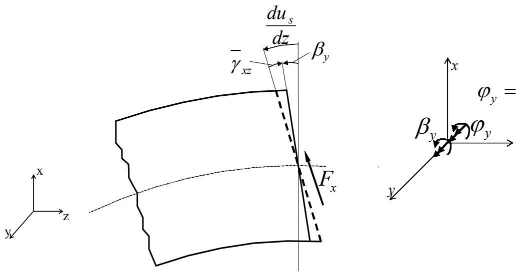

Fig. 1. Shear deformation in x-z plane

Flexible shear coefficients can be evaluated as:

$$\begin{aligned} &k_x = \frac{A}{I_y^2} \int_S C_y^2 \frac{dS}{dt}, \\ &k_y = \frac{A}{I_x^2} \int_S C_x^2 \frac{dS}{dt}, \\ &k_\omega = \frac{A}{I_\omega^2} \int_S C_\omega^2 \frac{dS}{dt} \end{aligned} \qquad (12)$$

where C_x, C_y and C_ω are first moments of area with respect to *x*, *y* and *ω* defined as:

$$\begin{aligned} &C_y = \int_A x \, dA = \int_0^S x t \, dS, \\ &C_x = \int_A y \, dA = \int_0^S y t \, dS, \\ &C_\omega = \int_A \omega \, dA = \int_0^S \omega t \, dS \end{aligned} \qquad (13)$$

2.2 Linearized virtual works principal

Proposing equilibrium between internal and external forces follows (Turkalj et.al., 2003 a, b):

$$\begin{aligned} &\int_V S_{ij} \, \delta e_{ij} \, dV + \int_V {}^0S_{ij} \, \delta\eta_{ij} \, dV + \int_V {}^0S_{ij} \, \delta\tilde{e}_{ij} \, dV - \\ &- \int_{A_\sigma} {}^0t_i \, \delta\tilde{u}_i \, dA_\sigma = \int_{A_\sigma} t_i \, \delta u_i \, dA_\sigma \end{aligned} \qquad (14)$$

Equation (14) is known as linearized principal of virtual works and it can be also rewritten as (Bathe, 1996):

$$\delta \mathrm{U}_E + \delta \mathrm{U}_G - \delta \mathrm{W} = \delta \Pi = 0 \qquad (15)$$

where elastic potential energy of internal forces is:

$$\delta \mathrm{U}_E = \int_V S_{ij} \, \delta e_{ij} \, dV,$$

geometric potential of initial forces is:

$$\delta \mathrm{U}_G = \int_V {}^0S_{ij} \, \delta\eta_{ij} \, dV + \int_V {}^0S_{ij} \, \delta\tilde{e}_{ij} \, dV - \int_{A_\sigma} {}^0t_i \, \delta\tilde{u}_i \, dA_\sigma,$$

virtual work of external forces is:

$$\delta \mathrm{W} = \int_{A_\sigma} t_i \, \delta u_i \, dA_\sigma$$

and Π is total potential energy.

Involving (1)-(6) into equations for $\delta \mathrm{U}_E$ and $\delta \mathrm{U}_G$ gives:

$$\delta \mathrm{U}_E = \int_0^l \left[EA \frac{dw_o}{dz} \delta \frac{dw_o}{dz} + EI_x \frac{d\varphi_x}{dz} \delta \frac{d\varphi_x}{dz} + EI_y \frac{d\varphi_y}{dz} \delta \frac{d\varphi_y}{dz} + \right.$$

$$+ EI_\omega \frac{d\theta}{dz} \delta \frac{d\theta}{dz} + GJ \frac{d\varphi_z}{dz} \delta \frac{d\varphi_z}{dz} + \frac{GA}{k_y}\left(\frac{dv_s}{dz} - \varphi_x \right)^2 +$$

$$\delta \mathrm{U}_G = \frac{1}{2}\int_0^l \Bigg\{ {}^0F_z \left[\delta\left(\frac{dw_o}{dz}\right)^2 + \delta\left(\frac{dv_s}{dz}\right)^2 + \delta\left(\frac{du_s}{dz}\right)^2 + \right.$$

$$\left. +2y_s\,\delta\left(\frac{du_s}{dz}\frac{d\varphi_z}{dz}\right) - 2x_s\,\delta\left(\frac{dv_s}{dz}\frac{d\varphi_z}{dz}\right)\right] +$$

$$+{}^0F_x\left[\delta\left(\varphi_x\varphi_z\right) + 2\delta\left(\frac{dv_s}{dz}\varphi_z\right) - 2\delta\left(\frac{dw_o}{dz}\varphi_y\right) + \right.$$

$$\left. +2x_s\,\delta\left(\frac{d\varphi_y}{dz}\varphi_y\right) - 2y_s\,\delta\left(\frac{d\varphi_x}{dz}\varphi_y\right)\right] +$$

$$+{}^0F_y\left[\delta\left(\varphi_y\varphi_z\right) - 2\delta\left(\frac{du_s}{dz}\varphi_z\right) + 2\delta\left(\frac{dw_o}{dz}\varphi_x\right) - \right.$$

$$\left. -2x_s\,\delta\left(\frac{d\varphi_y}{dz}\varphi_x\right) + 2y_s\,\delta\left(\frac{d\varphi_x}{dz}\varphi_x\right)\right] +$$

$$+{}^0M_z\left[\delta\left(\frac{d\varphi_x}{dz}\varphi_y\right) - \delta\left(\frac{d\varphi_y}{dz}\varphi_x\right)\right] +$$

$$+{}^0M_x\left[\delta\left(\frac{d\varphi_y}{dz}\varphi_z\right) + \delta\left(\frac{d\varphi_z}{dz}\varphi_y\right) - 2\delta\left(\frac{du_s}{dz}\frac{d\varphi_z}{dz}\right) + \right.$$

$$\left. +2\delta\left(\frac{dw_o}{dz}\frac{d\varphi_x}{dz}\right)\right] +$$

$$+{}^0M_y\left[-\delta\left(\frac{d\varphi_x}{dz}\varphi_z\right) - \delta\left(\frac{d\varphi_z}{dz}\varphi_x\right) - 2\delta\left(\frac{dv_s}{dz}\frac{d\varphi_z}{dz}\right) + \right.$$

$$\left. +2\delta\left(\frac{dw_o}{dz}\frac{d\varphi_y}{dz}\right)\right] +$$

$$\left. +{}^0\bar{K}\,\delta\left(\frac{d\varphi_z}{dz}\right)^2 + {}^0M_\omega\left[\delta\left(\frac{dw_o}{dz}\frac{d\theta}{dz}\right)\right]\right\}dz \tag{17}$$

3. THIN-WALLED BEAM FINITE ELEMENT

On the Fig. 2. the thin-walled beam finite element with 14 degrees of freedom is presented (Turkalj & Brnić, 2000; Turkalj et.al., 2003 c). The nodal displacement vector $\mathbf{u}^e$ and force vector $\mathbf{f}^e$ of an arbitrary *e*th element with the end nodes A and B, are:

$$\mathbf{u}^e = \begin{Bmatrix} \mathbf{u}_A^e \\ \mathbf{u}_B^e \end{Bmatrix}; \qquad \mathbf{f}^e = \begin{Bmatrix} \mathbf{f}_A^e \\ \mathbf{f}_B^e \end{Bmatrix}$$

or:

$$\left(\mathbf{u}^e\right)^T = \left\{ w_{oi} \quad u_{si} \quad v_{si} \quad \varphi_{zi} \quad \varphi_{xi} \quad \varphi_{yi} \quad \theta_i \right\}, \tag{18}$$

$$\left(\mathbf{f}^e\right)^T = \left\{ F_{zi} \quad F_{xi} \quad F_{yi} \quad M_{zi} \quad M_{xi} \quad M_{yi} \quad M_{\omega i} \right\} \tag{19}$$

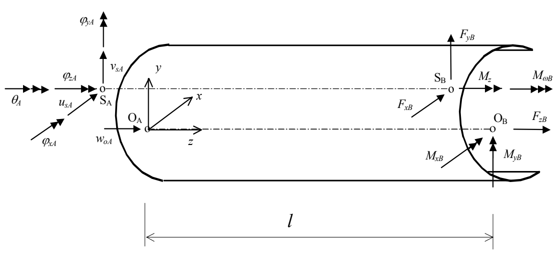

Fig. 2. Thin-walled beam finite element

$$+\frac{GA}{k_x}\left(\frac{du_s}{dz}+\varphi_y\right)^2 + \frac{GJ}{k_\omega}\left(\frac{d\varphi_z}{dz}+\theta\right)^2\Bigg]dz \tag{16}$$

For one finite element follows equilibrium equation:

$$\left(\mathbf{k}_E^e + \mathbf{k}_G^e\right)\mathbf{u}^e = \mathbf{f}^e\,. \tag{20}$$

where $\mathbf{k}_E^e$ and $\mathbf{k}_G^e$ are elastic and geometric parts of stiffness matrix in local coordinate system which are obtained by solving integrals for $\delta \mathrm{U}_E$ and $\delta \mathrm{U}_G$ (Turkalj et.al., 2002).

Interpolation functions for displacement components **w** are linear and for **u**, **v** and **φ** displacement is used cubic interpolation.

For the whole construction equilibrium equation using assumption of proportionality of loading is:

$$\left(\mathbf{K}_E + \lambda\hat{\mathbf{K}}_G\right)\mathbf{U} = \mathbf{F}\,, \tag{21}$$

where $\mathbf{K}_E$ is elastic stiffness matrix of construction, $\mathbf{K}_G$ is geometric stiffness matrix of construction, **U** and **F** are vectors of incremental nodal displacements and nodal forces of construction and λ is load parameter. Eigenvalues of equation (21) $\lambda_1,\ldots,\lambda_n$, presents critical buckling loads. Only the first value λ_1 is of practical interest (Mihanović, 1993).

4. EXAMPLE

Computer program THINWALL-SHEAR developed on the basis of the presented theory is tested on a two examples.

4.1 Torsional-flexural buckling of cantilever

Cantilever of unsymmetrical cross section, loaded with axial force at centroid is shown on Fig. 3.

Material and geometrical parameters are:

length	$l = 200$ cm;
modulus of elasticity	$E = 30000$ Ncm^{-2};
shear modulus	$G = 11500$ Ncm^{-2};
shear center coordinates:	$x_s = 1.58943$ cm;
	$y_s = -2.57228$ cm;
moments of inertia:	$I_x = 114.812$ cm^4,
	$I_y = 7.6048$ cm^4;
warping moment of inertia	$I_\omega = 70.9687$ cm^6;
torsional moment of inertia	$J = 0.666667$ cm^4;
shear factors:	$k_x = 5.2339$,
	$k_y = 1.794438$,
	$k_\omega = 0.01699$;
Wagner coefficients:	$\alpha_x = 5.66166$ cm,
	$\alpha_y = 11.0599$cm,
	$\alpha_z = 24.445$ cm^2,
	$\alpha_\omega = -0.558603$.

Values for critical buckling load F_{kr} evaluated by computer program THINWALL-SHEAR are compared with beam finite element results of (Kim et.al., 1994), and ABAQUS (Kim et.al., 2001) results who idealized cantilever with 1600 shell finite elements.

Table 1 shows very good accuracy of this paper results comparing with results of the others.

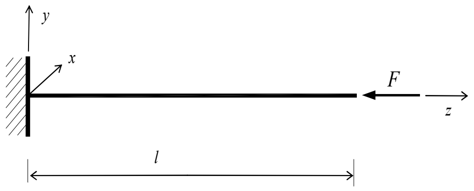

Fig. 3. Axially compressed cantilever of unsymmetric cross section from example 4.1

No. elem.	This paper	Kim *et.al.*	ABAQUS (*shell*)
1	13.9958		
2	13.8986	13.9017	14.0230
4	13.8930		

Table 1. Values F_{kr} (N) for cantilever from example 4.1

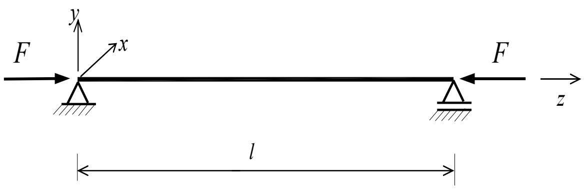

Fig. 4. Axially compressed simple beam of doubly symmetric cross-section from example 4.2

4.2 Flexural buckling of simply supported beam

Simply supported beam has doubly symmetric cross section The beam is axially loaded with axial force F . Due to doubly symmetry of cross section the only possible buckling mode is flexural.

The relevant material and geometrical properties for beam are:

length $l = 100$ cm;
modulus of elasticity $E = 2{,}1 \cdot 10^7$ Ncm^{-2};
shear modulus $G = 80{,}77 \cdot 10^5$ Ncm^{-2};
moments of inertia: $I_x = I_y = 50$ cm^4;
cross section area $A = 5$ cm^2.

In table 2. computer program THINWALL-SHEAR results are compared with numerical results, and Timoshenko analytical results by (Kim et.al., 1994) for different values of shear coefficients $k_x = k_y$.

From the table 2. for all range of values for shear coefficients k_x a very good coincidence of results can be seen.

$\frac{k_x}{GA} F_{kr}$	This work		Analytic	Kim *et.al.*
	No. elem.			
	2	4		
0	1.04410	1.03684	1.03630	1.03641
0,5	0.71541	0.69059	0.69087	0.69440
1	0.53516	0.51548	0.51815	0.52112
5	0.17193	0.16879	0.17272	0.17326
10	0.09254	0.09156	0.09421	0.09439
100	0.00992	0.00999	0.01026	0.01026
10000	0.00010	0.00010	0.00010	0.00010

Table 2. Values F_{kr} (N) for flexural buckling of beam under axial load ($\cdot 10^6$N)

5. CONCLUSION

Numerical models are only successful and cost low alternatives for analytical modeling which are applicable only in the case of very simple geometry of structure and for experimental observations of stability of real constructions which are of the best accuracy but of the highest price.

Presented numerical algorithm, based on the finite element method, includes large displacements and large rotations and also cross sectional deformation effects. Coincidence of results gotten by testing of developed computer program THINWALL-SHEAR on the few typical examples, with results of other authors available from literature, guarantees his successful appliance to the other problems of more complex constructions.

6. REFERENCES

Bathe, K. J. (1996). *Finite Element Procedure*, Prentice Hall, ISBN 0-13-349679-X, London

Chang, S. P., Kim, S. B., Kim, M. Y., (1996). Stability of Shear Deformable Thin-Walled Space Frames and Circulr Arches, *Journal of Engineering Mechanic*, Vol. 122, No. 9, (September 1996), pp. 844-854, ISSN 0733-9399

Kim, M. Y., Chang, S. P., Kim, S. B., (1994). Spatial Stability and Free Vibration of Shear Flexible Thin-Walled Elastic Beams. II: Numerical Approach, *Int. Journal for Numerical Methods in Engineering*, 37(1994), pp. 4117-4140, ISSN 0029-5981

Kim, M. Y., Chang, S. P., Kim, S. B., (2001). Spatial Posbuckling Analysis of Thin-Walled Frames. II: Geometrically Nonlinear FE Procedures, *Journal of Engineering Mechanics*, Vol. 127, No. 8, (August 2001), pp. 779-790, ISSN 0733-9399

Mihanović, A. (1993). *Stabilnost Konstrukcija*, Hrvatsko društvo građevinskih konstruktora, ISBN 953-6175-03-7, Zagreb, in Croatian

Turkalj, G., Brnić, J. (2000). Analiza elastičnog izvijanja tankostijenih grednih konstrukcija s obzirom na velike rotacije, *Strojarstvo*, 42 (5, 6) 2000, pp. 217-230, ISSN 0562-1887, in Croatian

Turkalj, G., Brnić, J., Prpić-Orsić, J. (2002). External Stiffnes Approach for Thin-Walled Frames with Elastic-Plasticity, Proceedings of the Sixth International Conference Computational Structures Technology, B.H.V. Topping and Z. Bittnar (Ed.), pp. 67-68, ISBN 0-948749-81-4, Prague-Czech Republic, September 2002, Civil-Comp Press, Stirling

Turkalj, G., Brnić, J., Lanc, D. (2003 a). Non-Linear Formulation for Elastic Stability Analysis of Thin-Walled Beam-Type Structures, In *Metal Structures: Design, Fabrication, Economy*, Jármai, K. and Farkas, J. (Ed.), pp. 79-86, Millpress, ISBN 90 77017 75 5, Rotterdam

Turkalj, G., Brnić, J., Lanc, D. (2003 b). Elastic-plastic large displacement analysis of thin-wallwd beam type structures, In *System-based Vision for Strategic and Creative Design,* Bontempi, F. (Ed.), pp. 639-644, A.A. Balkema Publishers, ISBN 90 5809 599 1, Lisse

Turkalj, G., Brnić, J., Prpić-Orsić, J. (2003 c). Large rotation analysis of elastic thin-walled beam-type structures using ESA approach, *Computers & Structures*, 81(2003), pp. 1851-1864, ISSN 0045-7949

Yang, Y. B. and Kuo, S. R. (1994). *Theory & Analysis of Nonlinear Framed Structures*, Prentice Hall, ISBN 0-13-109224-3, New York