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SIMULTANEOUS COLLISIONS OF RIGID BODIES IN MORE THAN ONE POINT

Janis Viba, Svetlana Polukoshko
Riga Technical University
Institute of Mechanics
Ezermalas Street 6, Riga, Latvia, LV-1006
pol.svet@inbox.lv

Abstract: In this paper we consider the multi-contacts collision of rigid bodies, in other words - simultaneous impact of solids in two or more points. It is difficulty to predict the result of such collision and behavior of bodies, especially for collisions with friction. Bodies can separate in contact point at the very beginning of impact, or at the end of impact, can slide during all impact period, or during part of the impact time. It is necessary to detect impulse forces for each contact point, and in the most case problem became unsolved. In this paper some cases of multi-contact collisions are discussed and some simple problems of such impact are solved in the framework of rigid body model, using impulse - momentum based approach.
Keywords: impact with friction, sliding, stick, separation, unilateral constraints.

1.INTRODUCTION

Multi-contact impact occurs when one moving body collides with fixed or moving another rigid bodies, most often such impact accompanies rolling and spherical motion of rigid bodies (Viba & Polukoshko, 2002). Simultaneous collisions of rigid body takes place even if some contact points have zero velocity, and only one non-zero. It is considered that contact points with zero velocity can transfer normal impact impulse (Wösle & Pfeiffer, 1996), collisions in every points start and come to the end simultaneously (Viba et al., 1998). Distinctive feature of rigid body motion with impacts is availability of unilateral constrains, and the number of equations of motion depends on state of the contacts. Thus, every closed contact reduces degree of freedom, tangential friction force changes stepwise. State of contact – separation, non-separation, sticking, sliding or reversed sliding – changes during time interval of impact (Plavnieks, 1969, Glocker, 1995), transition from non-separation to separation, from stick to sliding, from sliding to stick, from sliding in one direction to reversion sliding must be predicted in every case of impact. Multi-contacts impact problem is more complicated than single point impact because of large number of constraints.
In this study the problem is solved in the framework of rigid body, laws of impact is modeled after Poison, friction is taken into account after Coulomb with dry friction factor, deformations during collisions does not taken into account. If the short time of impact is divided into two phase– compression phase and restitution phase – it could be considered that inelastic collision has the first phase only. Traditional impulse - momentum based rigid body approach allows to write and computer symbolic math allows to solve the systems of equations not only for each phase of impact, but also for separate stages of each phase (Plavnieks, 1969).

1.TWO-POINTS COLLISIONS OF ROLLING CYLINDRICAL BODY

Rolling motion of cylindrical or spherical bodies on a surface with irregularity often accompanies by two-points collisions, which are considered to be simultaneous (Fig.1).

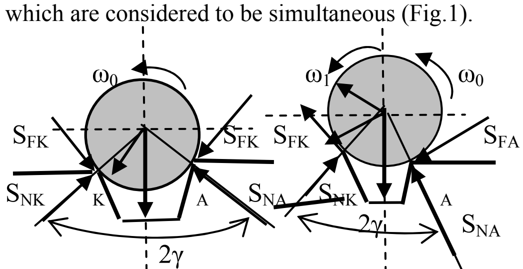

Fig. 1. **Rolling of the cylinder with impact in two points.**

Contact point ***K*** of body has both normal and tangential velocity, contact point ***A*** has only tangential velocity or zero velocity, if body rolls without sliding. Therefore, oblique central impacts can be only of two types: complete sliding in one direction in both point and termination of sliding in the first phase. If the first phase of impact or for all period of inelastic non-elastic following probable states of constraints can exist: 1- separation of a point ***A***, non-separation ***K*** with termination of sliding, 2 - separation of a point ***A***, non-separation of ***K*** with continuation of sliding, 3 - non-separation of a point ***A***, non-separation ***K*** with continuation of sliding and rotation of a body around of the center of mass, 4 - non-separation of a point ***A***, non- separation ***K*** with termination of sliding in the first phase and interruption of motion. For revealing conditions of separation - non-separation in the beginning of impact the basic equations of dynamics for impact are used, example of them with solving are given for the case 3 :

$$\text{Given}\quad m\cdot\omega 0\cdot a\cdot\cos(2\cdot\gamma) = -SNK\cdot f - SNA\cdot f\cdot\cos(2\cdot\gamma) - SNA\cdot\sin(2\cdot\gamma)$$

$$m\cdot\omega 0\cdot a\cdot\sin(2\cdot\gamma) = SNK + SNA\cdot\cos(2\cdot\gamma) - SNA\cdot f\cdot\sin(2\cdot\gamma)$$

$$Jc\cdot\omega 0 - Jc\cdot\omega 1 = SNK\cdot f\cdot a + SNA\cdot f\cdot a \qquad Jc = m\cdot a^2\cdot b$$

$$\text{Find}(SNA, SNK, \omega 1, Jc) \rightarrow \begin{bmatrix} -m\cdot\omega 0\cdot a\cdot\dfrac{(\cos(2\cdot\gamma) + f\cdot\sin(2\cdot\gamma))}{\sin(2\cdot\gamma)\cdot(f^2+1)} \\ m\cdot\omega 0\cdot a\cdot\dfrac{(\sin(2\cdot\gamma)^2 + \cos(2\cdot\gamma)^2)}{\sin(2\cdot\gamma)\cdot(f^2+1)} \\ \omega 0\cdot\dfrac{(-f + f\cdot\cos(2\cdot\gamma) + f^2\cdot\sin(2\cdot\gamma) + b\cdot\sin(2\cdot\gamma)\cdot f^2 + b\cdot\sin(2\cdot\gamma))}{b\cdot\sin(2\cdot\gamma)\cdot(f^2+1)} \\ m\cdot a^2\cdot b \end{bmatrix}$$

where: ***a*** - radius of a body, ***m*** – its mass, ***Jc*** - moment of inertia regarding to the center of mass, ***f***- factor of friction, ***2γ*** - central angle between ***A*** and ***K*** , factor ***b = Jc/m a²***.
It is evident from the solution, that normal impulse in point ***K*** is always positive and this point can't separate in the beginning of impact, but normal impulse in point ***A*** can be both positive and negative and point can separate dependent on ***2γ*** and ***f*** value.
The realization of the appropriate condition depends on motion before impact (with sliding or without it), from factor of instant friction at impact ***f***, angle ***2γ***, factor ***b***. The areas of existence of possible states with respect to angle ***γ*** and factor of friction ***f*** if pre-impact motion occurs without sliding for cylindrical body with ***b = 1*** are presented on the diagrams fig.2.

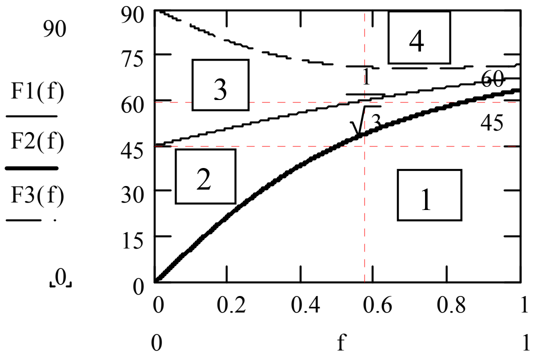

Fig. 2. **The diagrams of dependence of angle *γ* from factor of friction *f* for a body with *b = 1***

1-separation of point ***A***, impact is non-sliding
2- separation of point ***A***, impact is sliding
3- non-separation of point ***A***, impact is sliding
4 –non-separation of point ***A***, impact is non-sliding

If the impact is elastic there is the second phase of impact and possible state of constraints can be by the end of impact:
5- free plane motion without constraint after the first and the second mode of first phase of impact; 6 - rotation round point ***A*** without sliding after 3-rd mode of first phase, 7 – rotation round point ***A*** without sliding 4-th mode of first phase. Post-impact parameters may be defined from impulse-momentum equation if the coefficient of restitution is known.

3. TWO-CONTACTS COLLISIONS OF ROLLING POLYHEDROGONAL PRISM

When a homogeneous right-angle prism is rolling on smooth surface (Fig. 3) two-contacts impact is observed assuming that in the situation of balance prism rests upon a surface by two edges (further - points ***A*** and ***K***).

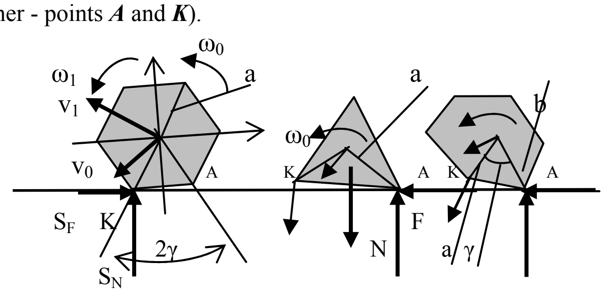

Fig.3 **Rolling of polyhedron prisms with impact in two points**

Prism collides with a plane in two points ***K*** and ***A***, further behavior of it may be: 1- no one point separate from surface, impact is non-sliding, prism motion stops; 2 - no one point separate from surface, impact is sliding, further motion – sliding on surface; 3 - point ***A*** separate from surface in the beginning of impact, point ***K*** sticks to surface, impact is non-elastic and non-sliding, further motion – rotation round point ***K***; 4 - point ***A*** separate from surface, impact is sliding, non-elastic, point ***K*** slips during impact, with further plane motion of prism; 5- point ***A*** separate from surface, impact is elastic, non- sliding, point ***K*** slips during impact and rebounds at the end of it, with further plane motion; 6) the same as previous case - point ***A*** separate from surface, impact is elastic, sliding, point ***K*** rebounds at the end of impact, result is plane motion; 7) - point ***A*** does not separate from surface, impact is elastic, sliding, point ***K*** rebounds at the end of impact, result is plane motion; 8) point ***A*** does not separate from surface, impact is elastic, non-sliding, point ***K*** rebounds at the end of impact, result is rotation motion round point ***A***.
Impulse-momentum equations for the first phase of impact gives normal impulses in these points for regular prism subject to ω_1=0:

$$S_{NK} = \omega_0 \cdot \frac{Jc + m \cdot a^2}{2 \cdot a \cdot \sin\left(\frac{\pi}{n}\right)} \qquad S_{NA} = \omega_0 \cdot \frac{2 \cdot m \cdot a^2 \cdot \sin\left(\frac{\pi}{n}\right)^2 - \left(Jc + m \cdot a^2\right)}{2 \cdot a \cdot \sin\left(\frac{\pi}{n}\right)}$$

Jc- moment of inertia with regard to central axis, ***n*** –number of side of regular prism, ***m*** - mass of prism.
Impact will be non-sliding if friction factor is:

$$f \geq \frac{1}{\tan\left(\frac{\pi}{n}\right)}.$$

Point ***K*** never separates from a surface in the beginning of impact, point ***A*** does not separate only in case of a triangular prism both for sliding and non-sliding impacts.
For irregular prism point ***A*** does not separate under condition of:

$$Jc \leq m \cdot a \cdot b \cdot \cos\left(\gamma_i\right)$$

and if central angle γ_i is obtuse.
After separation of a point ***A*** we can easy determine post-impact velocities using general equations of dynamics for impact and predict post-collision motion.

4. TWO-POINTS COLLISIONS OF ROTATION RIGID ROD

The investigated rod is thin enough and tangential impulse force may be neglected; sliding impact is excluded. Rod is turn round the support ***A*** on initial angle ***α*** (Fig. 4), then released and rotates under gravity round fixed support ***A***. When the rod reaches fixed support ***B***, angular rate of it is equal:

$$\omega_0 = \sqrt{\frac{2 \cdot m \cdot g \cdot a}{Jc + m \cdot a^2} \cdot \sin(\alpha)} = \sqrt{\frac{24 \cdot g \cdot a}{l^2 + 12 \cdot a^2} \cdot \sin(\alpha)},$$

where ***Jc =ml²/12*** –moment of inertia with regard to axis passed through mass center, ***l*** - length of rod, ***m*** –rod mass.
In this moment two-contacts collision (in points ***A*** and ***B***) is beginning,
The investigation of collision shows that there are four possible variants of the rod behavior: 1- the rod doesn't separate from supports and the motion ends – the collision is perfectly inelastic; 2- separation of the rod from support ***A*** at the beginning of impact and further rotation round point ***B*, ** in this case the collision is perfectly inelastic; 3- rod bounces off from support ***B*** at the end of impact and does not separate or rebound from support ***A*** with further rotation round point ***A***, it means the restitution of normal impulse in the point ***B*** and elastic collision;

4 - rod separates from *A* at the beginning of the collision and bounces off from support *B* at the end of the second phase with unrestricted plane motion under gravity, the collision is elastic.
The motion between impacts is described by the differential equation for perfectly rigid body, the impact is described by principle of momentum and principle of moment of momentum of rigid body for both phases of impacts in combination with Poisson impact law.
In order to determine the condition of the sticking of rod to the supports or separation from it rigid body momentum theorem in projection on ***y*** axes (3) and principle of moment of momentum in regard to mass center (4) is drown up for the first phase of impact, taken into account normal reactive impulses in points ***A*** and ***B***, and is solve with help of MATHCAD-2001. It is assumed that rod separates from support when calculated from the system of the general dynamics equations normal reactive impulse force is negative or equal zero, if the rod sticks to both support it angular speed equal zero.

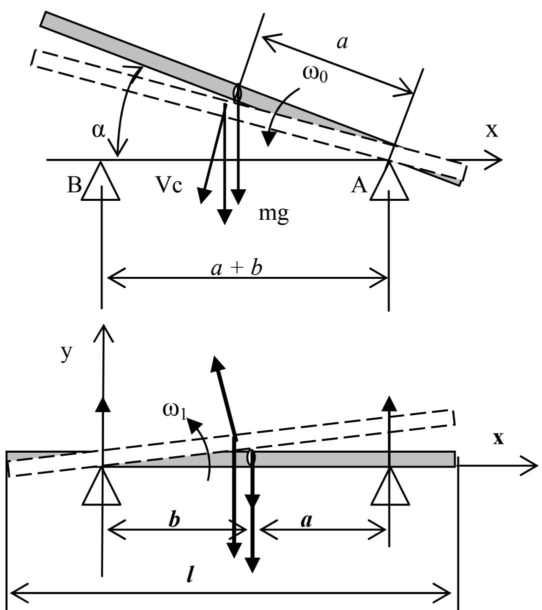

Fig.4. **Rotation motion of rod with impacts against fixed supports.**

$$\text{Given}\quad m\cdot Vcy1 - (-m\cdot Vcy0) = SYA + SYB \qquad Jc = \frac{m\cdot l^2}{12}$$

$$Jc\cdot\omega 1 - Jc\cdot\omega 0 = SYA\cdot a - SYB\cdot b$$

$$Vcy1 = \omega 1\cdot b \qquad Vcy0 = \omega 0\cdot a \qquad \omega 1 = 0$$

$$\text{find}(SYA, SYB, \omega 1, Jc, Vcy1, Vcy0) \rightarrow \begin{bmatrix} \frac{1}{12}\cdot m\cdot\omega 0\cdot\frac{\left(12\cdot a\cdot b - l^2\right)}{(a+b)} \\ \frac{1}{12}\cdot m\cdot\omega 0\cdot\frac{\left(l^2 + 12\cdot a^2\right)}{(a+b)} \\ 0 \\ \frac{1}{12}\cdot m\cdot l^2 \\ 0 \\ \omega 0\cdot a \end{bmatrix}$$

The solving of this system shows that at the beginning of the first case the rod can't rebound from support ***B***, and don't separate from ***A*** support only subject to: $\boldsymbol{12\cdot a\cdot b > l^2}$, if ***a=b***, this condition is $\boldsymbol{a > 0.289\cdot l}$. If there is no the second phase of impact (impact is perfectly inelastic), after separating from support ***A*** and energy and angular speed losing during impact, the rod goes on rotation round support ***B*** with angular speed:

$$\omega 1 = \omega 0\cdot\frac{l^2 - 12\cdot a\cdot b}{l^2 + 12\cdot b^2} \qquad \omega 1 = \omega 0\cdot k \qquad k = \frac{l^2 - 12\cdot a\cdot b}{l^2 + 12\cdot b^2}$$

This results was examined on experimental set and was entirely confirmed. Only linear dimensions of the rod – length and distance between supports influence on the condition of separation or sticking and on the angular speed decrease coefficient, mass and speed don't influence them.
Rod cross-section dimensions influence on normal reaction restitution.

5. MULTI-CONTACTS IMPACT DURING SPHERICAL MOTION OF SOLIDS

5.1 Three-points collisions in course of spherical motion of pyramid solids

Motion of regular pyramid on a fixed plane, i.e. rolling them around of a lateral edge accompanies with three-impacts of a pyramid about the surface (Fig. 5).

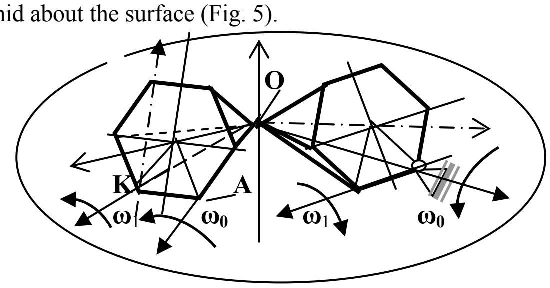

Fig. 5. **Rolling of a pyramids on a plane with three -contacts impact**

The impacts are considered to be perfectly inelastic; a pyramid turns round lateral edge without sliding; the speeds of points ***O*** and ***A*** at the beginning of impact are equal to zero, velocity of a point ***K*** is directed perpendicularly to plane, impact in a point ***K*** is direct, eccentric, there is tangential component of impact impulse and during impact a sliding can appear.
For examination the possibility of separation or non-separation contact points from a surface the general equations of dynamics for impact – momentum theorem and principle of moment of momentum in a projection to the appropriate coordinates axes are used. All variants of behavior of bodies are checked up at the instant of a beginning of impact - separation of all three points, non-separation of all points, non-separation of points ***K*** and ***A***, non-separation of ***A*** and ***O***, non-separation of ***K*** and ***O***, non-separation of ***K***, non-separation of ***A***, non-separation of ***O***, from which only two are possible - non-separation ***K*** with a separation ***A*** and ***O*** and non-separation ***K*** and ***O*** with a separation ***A***. The appearance of the appropriate variant depends on the geometrical characteristics of a homogeneous pyramid: its height ***h***, circumcircle radius ***a*** of the polygon base of pyramid and its number of the sides ***n***. Under condition of

$$h > a\cdot\sqrt{1 - \frac{2}{3}\cdot\sin\left(\frac{\pi}{n}\right)^2}$$

i.e. in case of a "long" pyramid, only the vertex of the basis A is separated, and the post-impact motion is rotary around of a lateral edge ***OK***, it determines a post-impact velocity vector direction; if this condition is not carried out, i.e. in a case of a "short" pyramid, the vertex ***A*** of the base is also separated and post-impact motion will be spherical with the centre in a point ***K***.

5.2 Multi-points collisions in course of spherical motion of symmetric solids

Occurring by gravity spherical motion of bodies located on a horizontal rough plane (Fig. 6) often comes to the multi-contacts collision. Homogeneous, symmetric bodies with regular, located on a circle of radius ***a*** supports, which number ***n*** is more or equally to three are considered here; height of all supports is identical, in a position of balance the body concerns a surface by all supports. The spherical motion of a body occurs round one of support without sliding and begins after taking body aside from a situation of balance; the initial conditions excludes the overturning of a body.

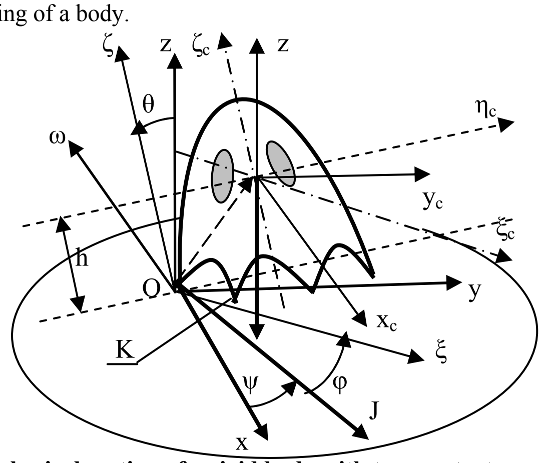

Fig. 6. **Spherical motion of a rigid body with two-contacts impact**

Position of a body is defined by Euler angles ψ, φ, θ; system of differential equation of motion is solved numerically and the moment of collision can be determined when one of the nearest to the point ***O*** support touches with surface. The body can touch a plane by all support at once, i.e. when the corner ***θ*** becomes equal 0, it takes place only at the certain initial conditions.
Depending on geometry of a body and initial conditions at the beginning of impact, non-separation from a surface of both points, separation only of points ***O*** or only of points ***K*** are possible. If no one point separates from a surface, resulting motion will be rotation and the vector of post-impact angular velocity is directed on a line ***OK*** and its module is equal:

$$\omega 1 = \frac{-J\xi \cdot \cos\left(\frac{\pi}{n}\right)\cdot \omega 0\xi + J\eta \cdot \sin\left(\frac{\pi}{n}\right)\cdot \omega 0\eta - J\eta\zeta \cdot \sin\left(\frac{\pi}{n}\right)\cdot \omega 0\zeta}{J\eta \cdot \sin\left(\frac{\pi}{n}\right)^2 + J\xi \cdot \cos\left(\frac{\pi}{n}\right)^2}$$

here $\omega_{o\xi}$, $\omega_{o\eta}$,$\omega_{o\zeta}$–projections onto moving axes ξ,η,ζ pre-impact instant angular velocity, $J\xi$, $J\eta$, $J\eta\zeta$ – moments and product of inertia.

7. CONCLUSIONS

Fruitful model of rigid body proves to be very useful in case of multi-contacts impact and allows to predict state of unilateral constraints. It is proved that approximate treatment of rigid body collision with Poisson impact law, which does not take into account deformation during collision, may be successfully used for determination of the uniform rigid bodies behaviour.

In this study all possible states of constraint are considered for each contact point, and all possible motion of rigid body, which constraints can allow are considered with different constraint state combinations, all possibilities are examined with computer algebra. In future it is supposed to apply this approach to spinning motion with impact in two points.

8. REFERENCES

Glocker Ch. (1995), *Dynamik von Starrkörpersystemen mit Reibung und Stößen.* Doktor-Ingenieurs Genehmigten Dissertation, München Technischen Universität, 1995

a. Glocker Ch. & Pfeiffer F. (1992). Dynamical systems with unilateral contacts. *Nonlinear Dynamics*, 9(3), 1992, pp. 245-259

b. Glocker Ch. & Pfeiffer F. (1994). Stick-Slip Phenomenon and Application. Nonlinearity and Chaos in Engineering Dynamics. *IUTAM Symposium,* UCL, July 1993, pp. 103–113. John Wiley & Song, 1994

Kawachi K., Suzuki H. & Kimura F. (1998). Technical Issue on Simulating Impulse and Friction in Three Dimensional Rigid Body Dynamics. *Proceeding of IEEE Computer Animation '98 Conference*, pp. 7-13, June, 1998.

Kononova O., Viba J., Polokoshko S.& Sokolova S. (2002). Symbolically Calculation of Collisions in Connected Body System. *Proceedings of the 3rd international DAAM conference INDUSTRIAL ENGINEERING – NEW CHALLENGES TO SME*, pp. 30-33, Tallinn, April 25-27, 2002. Estonia

Lavendelis E., Viba J. & Grasmanis B. (1997). Collision of the Rigid Body with Obstacle at More than One Point. *Proceedings of the 2nd International conference "MECHANICS",* Part 1, pp. 88-94, Vilnius, September 23-25, 1997, Lithuania

Nagaev R.F. (1985) *Механические процессы с повторными затухающими соударениями*. Nauka, Moscow

Plavnieks,V.(1969) Oblique Collision of Two Bodies. *Problems of Dynamics and Strength.* No 19, pp. 69-82. Zinatne, Riga

Viba J., Grasmanis B. & Fontaine J. (1998). Simultaneously Collisions in Connected Bodies Systems. *Proceedings of IUTAM/ IFToMM Symposium, Solid mechanic and its application.* Vol. 73, pp. 267-274. August 24-28, 1998. Riga, Latvia. Kluwer Academic Publishers. Dordrecht/ Boston, London.

Viba J. & Polokoshko S. (2002). Motion of Solids, Accompanied by Serial Collisions. *Proceedings of the 3rd international DAAM conference INDUSTRIAL ENGINEERING – NEW CHALLENGES TO SME*, pp. 70-73, Tallinn, April 25-27, 2002. Estonia

Wösle, M. & Pfeiffer, F. (1996). Dynamics of Multibody Systems Containing Dependent Unilateral Constraints with Friction. *Lehrstuhl B für Mechanik, Technische Universität München, Germany.* Available from: http://www.lbm.mw.tu-muenchen.de/Research/Publikationen/woesle/jovac.ps.gz