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IMPACT SYSTEM RESTITUTION COEFFICIENT DEPENDENCE ON GEOMETRY

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Abstract: *One of the important parameter of impact process is the coefficient of restitution. Practically important task is to determine the coefficient of restitution in a contact tasks, when is necessary to avoid repeated transient processes during the contact. Many researches accept the coefficient of restitution a constant (Viba, 1988) (Plavnieks, 1969). Contrary in present report this coefficient is obtaining during calculations. Our aim is to obtain an effect equal to plastic impact, when not repeated impacts will happened in system. Prepared calculations shown detailed wave interaction picture in construction. Comprehensive parametric analysis was realized. Results of elastic and viscoelastic material behaviour were compared.*
Key words: coefficient of restitution, impact, finite differences, distributed parameters, boundary conditions.

1. INTRODUCTION

During the impact in complex beam constructions occur both longitudinal and transverse waves of deformations. In this report only transverse waves of deformation will be considered. The beam is the system with distributed parameters, where the rigidity and the mass distributed uniformly along the beam. (Viba, et al., 2000) The fixed base is absolutely rigid. One of the most important tasks is the determination of time of impact, moment when the beam rebounds from fixed base. For description of beam behaviour during impact are some methods. These methods will be observed in this paper.

2. CLASSICAL EQUATION OF STRENGTH OF MATERIALS

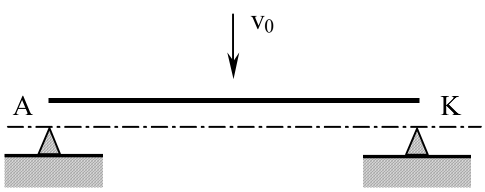

Fig. 1. Model of plane impact of system with distributed parameters

2.1. Exact solution (Fourier's method)

The plane impact of beam in two points is shown in Fig. 1. The contact in points A and K continues during the impact. In the simplest case the beam behaviour during of impact can be described by differential equation (Timoshenko et al., 1985):

$$\frac{\partial^2 y}{\partial t^2} + \frac{EJ}{m}\frac{\partial^4 y}{\partial x^4} = 0, \qquad (1)$$

where: y – vertical displacement of the beam;
J – moment of inertia of the beam section;
m – intensity of the beam mass (mass of unit of length).
In order to solve equation (1) the Fourier's method was used. (Timoshenko et al., 1985) In this case vertical displacement y can be rewritten in the form:

$$y = \sum_{n=1}^{\infty} X_n(x) T_n(t), \qquad (2)$$

where: T_n – is fundamental function of time;
X_n – is fundamental function of displacement;
n – is number of beam frequency. Due to beam is the system with distributed parameters, it has some natural frequencies n. Boundary conditions for the beam are given by: vertical displacement of beam in contact points A and K is zero $y = XT$ and bending moment in same points is zero $M = EJX''T$. So, boundary conditions is

$$\begin{aligned} &X(x=0)=0;\\ &X''(x=0)=0;\\ &X(x=l)=0;\\ &X''(x=l)=0, \end{aligned} \qquad (3)$$

where x – is coordinate of beam section at horizontal direction;
l – is length of beam.
Fundamental function of displacement is

$$X_n = C_n \sin(k_n x) \qquad (4)$$

where C_n – is constant depending on scale;
k is given by

$$k = \sqrt[4]{\frac{mp^2}{EJ}} \qquad (5)$$

where p - is natural frequency of beam

$$p_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EJ}{m}} \qquad (6)$$

The initial conditions for our case are initial displacement and initial velocity:

$$\left.\begin{aligned} &y(x,0) = y_0(x);\\ &v(x,0) = \dot{y}(x,0) = v_0(x). \end{aligned}\right\} \qquad (7)$$

So, general solution can be rewritten in the form:

$$y = \sum_{n=1}^{\infty} X_n(x)(A_n \sin(p_n t) + B_n \cos(p_n t)). \qquad (8)$$

where constants A_n and B_n are equals:

$$\left.\begin{aligned} A_m &= \frac{\int_0^l v_0(x) X_m(x)dx}{p_m \int_0^l [X_m(x)]^2 dx};\\ B_m &= \frac{\int_0^l y_0(x) X_m(x)dx}{\int_0^l [X_m(x)]^2 dx}. \end{aligned}\right\} \qquad (9)$$

For our case constants are equals:

$$\left.\begin{aligned} A_m &= \frac{2v_0(1-\cos(k_n l))}{C_n p_m k_n l}; \\ B_m &= 0. \end{aligned}\right\} \qquad (10)$$

General solution is

$$y = \sum_{n=1}^{\infty} \sin(k_n x) \frac{2v_0(1-\cos(k_n l))}{p_n k_n l} \sin(p_n t) \quad (11)$$

2.2. Method of finite differences

In order to solve the differential equation (1) the method of finite differences was used (Krylov et al., 1977). Then the differential equation (1) becomes:

$$\begin{aligned} &\frac{1}{\tau^2}(y_{t+1,x} - 2y_{t,x} + y_{t-1,x}) + \\ &+ \frac{EJ}{mh^4}(y_{t,x+2} - 4y_{t,x+1} + 6y_{t,x} - 4y_{t,x-1} + y_{t,x-2}) = 0 \end{aligned} \qquad (12)$$

Boundary and initial conditions are the same as previous case. By using the finite differences, boundary conditions can be rewritten in the form:

$$\begin{aligned} &y_{t,0} = 0 \\ &y_{t,L} = 0 \\ &\frac{1}{h^2}(y_{t,0+h} - 2y_{t,0} + y_{t,0-h}) = 0 \\ &\frac{1}{h^2}(y_{t,L+h} - 2y_{t,L} + y_{t,L-h}) = 0 \end{aligned} \qquad (13)$$

By using the finite differences, initial conditions can be rewritten in the form:

$$\begin{aligned} &y_{0,x} = 0 \\ &\frac{1}{\tau}(y_{\tau,x} - y_{0,x}) = v_0 \end{aligned} \qquad (14)$$

where $x = 0 \ldots L$, h – integration step in horizontal direction (x-direction), $t = 0 \ldots T$, τ - – integration step in t-direction (time direction).

2.3. Comparison of Fourier's method and method of finite differences

Exact solution (Fourier's method) was realized by using of program MathCAD, the method of finite differences by using of program MATLAB as it shown in Fig.2. Number of sum for exact solution n is equal 5 (See equation 2).

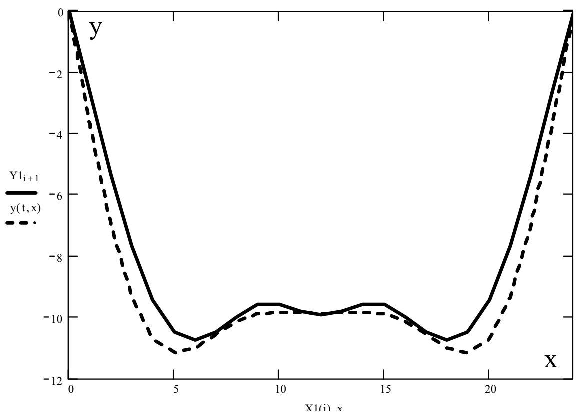

Fig. 2. Distribution of vertical displacements of beam points during the impact.

______ - method of finite differences

_ _ _ _ - Fourier's method.

For determination of coefficient of restitution we must know the moment of end of impact. At the end of impact the cross force in the point of fixed base change the sign and the beam rebounds from fixed base. In the Fig.3 is shown the dependence of value of cross force on time in the case of exact solution (Fourier's method). You can see that value of cross force in fixed base is depending on number of sum. The cross force in this case is from equation (10)

$$\begin{aligned} q(t,x) &= \sum_{n=1}^{N} k_n{}^3 \cos(k_n x) \times \\ &\times \frac{2v_0(1-\cos(k_n L))}{p_n k_n L} \sin(p_n t) \end{aligned} \qquad (15)$$

where N – is number of sum;

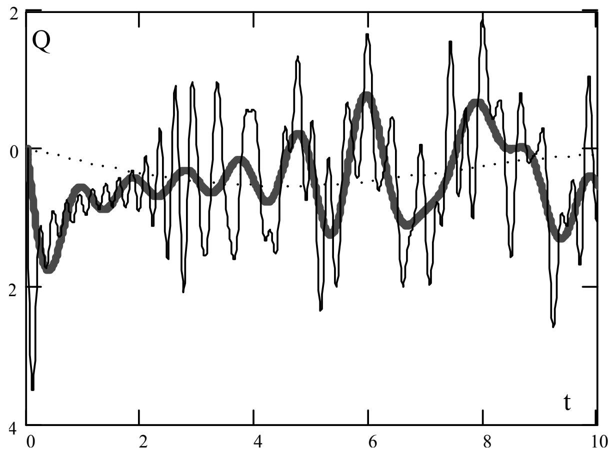

Fig. 3. The cross force depending on time in case of Fourier's method.

— — — - number of sum N=4;

▬▬▬ - number of sum N=10;

———— - number of sum N=20.

Similar figure will be if we use the classical equation of strength of material. In this case the moment of end of impact is depending on integration step in horizontal direction (x-direction). This case is shown in Fig. 4.

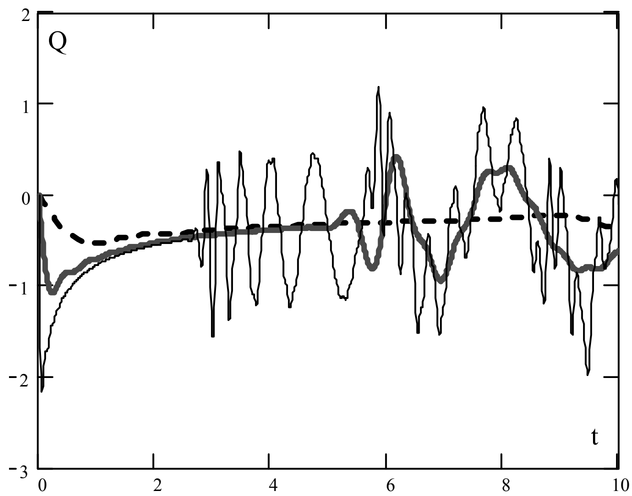

Fig. 4. The cross force depending on time in case of classical equation of strength of material.

— — — – integration step in horizontal direction is h;

▬▬▬ – integration step in horizontal direction is h/2;

———— – integration step in horizontal direction is h/4.

The classical equation of strength of material and Fourier's method not take into consideration inertia of rotation and cross displacement, because in the real impact system cross-sections of beam can't realize the high frequencies.

3. CLASSICAL EQUATION OF STRENGTH OF MATERIAL WITH VISCOUS DAMPING

For determine the coefficient of restitution of beam during the impact we must to use the other differential equation. In the case of taking into consideration viscous damping for equation (1) normal stress can be rewritten in the form:

$$\sigma = E\varepsilon + k\frac{d\varepsilon}{dt} \tag{16}$$

where k – is coefficient of viscous damping;
ε – is the deformation of beam element.
The deformation of beam element is

$$\varepsilon = -cy \tag{17}$$

where c –is coefficient of proportionality.

$$c = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}. \tag{18}$$

where Θ – is the angle of rotation of cross-section.
So normal stress can be rewritten in the form:

$$\sigma = -E\frac{\partial^2 y}{\partial x^2}y - k\left(\frac{\partial^3 y}{\partial x^2\partial t}y + \frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t}\right). \tag{19}$$

On the other hand bending momentum is

$$M = -\int_F y\sigma dF = = EJ\frac{\partial^2 y}{\partial x^2} + kJ\frac{\partial^3 y}{\partial x^2\partial t} + kS_x\frac{\partial^3 y}{\partial x^2\partial t} \tag{20}$$

where F – is square of beam cross-section;
J – is momentum of inertia of beam cross-section;
S_x – is static momentum of beam cross-section
But for symmetric section S_x=0 and distributed load q is

$$q = \frac{\partial^2 M}{\partial x^2} = -m\frac{\partial^2 y}{\partial t^2} \tag{21}$$

So differential equation (1) can be rewritten in the form:

$$EJ\frac{\partial^4 y}{\partial x^4} + kJ\frac{\partial^5 y}{\partial x^4\partial t} + m\frac{\partial^2 y}{\partial t^2} = 0 \tag{22}$$

But this model does not solve the problem of determination of time of impact, because in this case as a previous cases cross force is depending on integration step in horizontal direction.

4. TIMOSHENKO EQUATION

Previous methods of classical equations of strength of materials and Fourier's method can be used in the cases if size of beam cross-sections is small in comparison with length of beam. If length of beam is in proportion with cross-section we must use Timoshenko equation witch take into consideration inertia of rotation and cross displacement of sections (Timoshenko et al., 1985). Total angle of rotation of cross-section is equal:

$$\theta = \frac{dy}{dx} = \psi + \beta \tag{23}$$

where ψ – is the angle of rotation of cross-section;
β – is the angle of cross displacement.
For this case the bending momentum and cross force are:

$$M = EJ\frac{\partial\psi}{\partial x}$$
$$Q = -k'\beta FG = -k'\left(\frac{\partial y}{\partial x} - \psi\right)FG \tag{24}$$

where G –is cross module;
Q- is cross force.
k' - is the coefficient of form of cross-section (for the rectangular cross-section k'=5/6).
Differential equation of equilibrium of momentums for one element of beam is:

$$-Qdx + \frac{\partial M}{\partial x}dx - \rho J\frac{\partial^2\psi}{\partial t^2}dx = 0 \tag{25}$$

where ρ –is thickness of beam material.
By using the equations (24) last equation can be rewritten in the form:

$$EJ\frac{\partial^2\psi}{\partial^2 t} + k'\left(\frac{\partial y}{\partial x} - \psi\right)FG - \rho J\frac{\partial^2\psi}{\partial t^2} = 0 \tag{26}$$

Differential equation of equilibrium of forces for one element of beam is:

$$-\frac{\partial Q}{\partial x}dx - \rho F\frac{\partial^2 y}{\partial t^2}dx = 0 \tag{27}$$

By using the equations (24) last equation can be rewritten in the form:

$$k'\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right)G - \rho\frac{\partial^2 y}{\partial t^2} = 0 \tag{28}$$

For solve of system of equations were used boundary conditions (13), initial conditions (14) and equations (26), (28). For comparison of precision of all methods was used the real material - steel. In Fig. 5 is shown distribution of vertical displacements of beam points during the impact ($t=25*10^{-6}$ s).

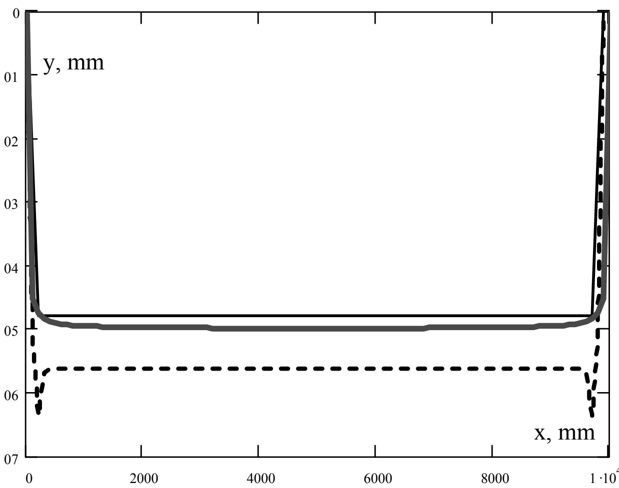

Fig. 5. Distribution of vertical displacements of beam points during the impact ($t=25*10^{-6}$ s). Material – steel, $E=2*10^{11}N/m^2$; $G=8*10^{10}$ N/m^2; $\rho=7,8*10^3$ kg/m^3; $a=2*10^{-2}m$; $b=1*10^{-2}$ m (sizes of cross-sections), l=2,5 m (length of beam).
– – – – Timoshenko equation (finite differences);
——— – classical equations of strength of materials (finite differences);
——— – Fourier's method.

The general advantage of Timoshenko equation is what this equation gives us the possibility to determine the time of end of impact. In the case if we use equations (26) and (28) cross force weakly depend on integration step of beam as it shown in Fig. 6. Dependences of cross forces on time were taken for the same material. Boundary conditions (14) take place only during the impact. The impact is finished at the moment when the cross force changes the sign. In Fig.6 this is the point of intersection of curve of cross force and straight line where Q=0. In this case exists the difference in point of end of impact by different integration steps, but this difference is much smaller then in case of Fourier's method or of strength of materials method (see Fig.3, Fig.4). Synthesis of different methods to let us to obtain the dependence of coefficient of restitution on geometry of impact system. For determination of time of impact was used Timoshenko method, for determination the velocities in this moment was used the classical method of strength of materials with viscous damping. In order to solve the equation (22), boundary conditions (13), initial conditions (14) the method of finite differences was used.

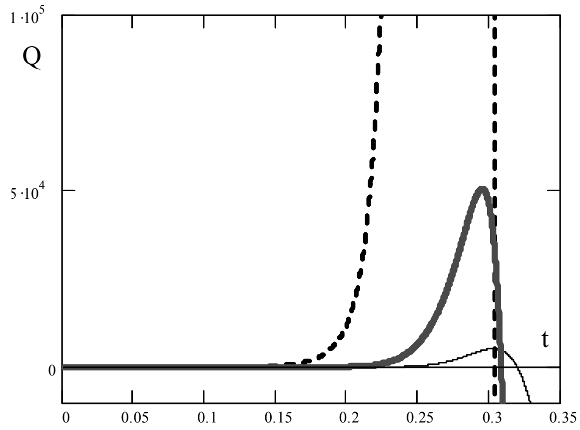

Fig. 6. The cross force depending on time in case of Timoshenko equation.

– integration step in horizontal direction is h=100mm;

– integration step in horizontal direction is h=150mm;

– integration step in horizontal direction is h=200mm.

In the case of plane impact (Fig.1.) the coefficient of restitution is ratio of impulse before and after the impact.

$$R = \frac{\sum m_k \dot{y}_k}{|Mv_0|} \tag{29}$$

where m_k – is the mass of one element of beam;
M- is the mass of beam;
v_0 – is the velocity of beam before impact;
$\dot{y}_k$ - is the velocity of one element of beam.

The calculations show that with increase of beam length the coefficient of restitution is decrease, it shown in Fig.7.

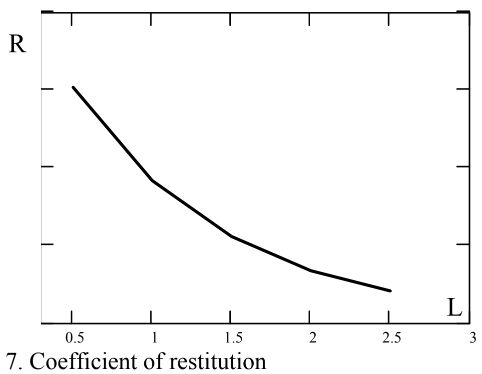

Fig. 7. Coefficient of restitution

5. CONCLUTIONS

Timoshenko method to let us to determine time of impact, the moment when beam rebounds from fixed base. Using Timoshenko equations and method of finite differences we may obtain only time of impact, but not the value of cross force. Because the value of force in the point of fixed base is depending on step horizontal direction as in similar models. Apparently this is due to the fact that the fixed base is absolutely rigid but the beam is system with distributed parameters.

Approximate calculation on basis of two models: Timoshenko method and classical equation of strength of material allow concluding what the coefficient of restitution decrease with increase of length of beam.

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