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NUMERICAL SOLUTION OF A CLASS OF FRACTIONAL DELAY DIFFERENTIAL EQUATIONS VIA HAAR WAVELET

Aziz, I; Amin, R & Majak, J.

Abstract: *In this paper, Haar wavelet collocation method is applied for the numerical solution of fractional delay differential equations. The method is applied to linear and nonlinear fractional delay differential equations. The numerical results are compared with the exact solutions and the performance of the method is demonstrated by calculating the maximum absolute errors and mean square roots errors for different number of collocation points. The numerical results show that the method is simply applicable, accurate, efficient and robust.*
Keywords: Fractional calculus, Caputo derivative, Haar wavelet, fractional delay differential equations.

1. INTRODUCTION

The subject of fractional calculus deals with generalizations of differentiation and integration of arbitrary orders and dates back to correspondence between L' Hospital and Leibniz towards the end of 17th century. This was followed by the contributions from Euler and Lagrange in 18th century. Abel solved the integral equations encountered in the tautochrone problems using fractional derivatives, a notion not so well formulated then. The work of Abel gave further stimulus to the development of the subject. The pioneering works of Liouville, Riemann, Grunwald and Letnikov in the middle of 19th century finally led to formulation of fractional integrals and derivatives with subsequent development of fractional calculus [1]. The fractional derivatives have less properties than the corresponding classical ones. As a result, it makes these derivatives very useful in describing anomalous phenomena [2]. Recently, an important attempt to give a physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives was suggested in [3].

Fractional calculus has been used to model physical and engineering processes that are found to be the best described by fractional differential equations. For that reason we need a reliable and efficient technique for the solution of fractional differential equations. Fractional delay differential equations have recently been applied in various areas of engineering, science, finance, applied mathematics, bioengineering and others [4].

The use of wavelets has come to prominence during the last two decades. They have wide- ranging applications in scientific computing, and it is no surprise that they have been extensively used in numerical approximation in the recent relevant literature. Some of the recent work using wavelets can be found in the references [5–7, 12-14].

In the present work, we will consider fractional delay differential equations with time-delay τ in the state of the following form [8]:

$$D^{\alpha}u(t) = a\,u(t-\tau) + b\,u(t) + f(t), t > 0,$$
$$u(t) = \varphi(t), t \in [-\tau, 0],$$
$$u(0) = u_0, \tag{1}$$

where $u(t)$ is the state function, $u(0)$ is the initial condition, a and b are constants, f is a continuous function on $[0, T], T > 0, \varphi(t)$ is the delay condition continuous on $[-\tau, 0]$, and D^{α} is the fractional derivative of order α which will be considered in the Caputo sense in this paper.

The Caputo fractional derivative operator D^{α} of order α was introduced by M. Caputo in 1967 and is defined as [9]:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)dt}{(x-t)^{1+\alpha-n}}, \alpha > 0, \tag{2}$$

where $n-1 < \alpha < n, n \in \mathbb{N}, x > 0$.

The paper is organized in the following structure. In Section 2, Haar wavelet is introduced. The numerical method for the solution of fractional delay differential equations based on the Haar wavelet is developed in Section 3. In Section 4, numerical experiments are performed. Finally, some conclusions are drawn in Section 5.

2. HAAR WAVELET

The Haar wavelet family for $x \in [0,1)$is defined as [5]:

$$h_i(t) = \begin{cases} 1 & t \in [\xi_1, \xi_2), \\ -1 & t \in [\xi_2, \xi_3), \\ 0 & \text{elsewhere}, \end{cases} \tag{3}$$

where

$$\xi_1 = \frac{k}{m}, \xi_1 = \frac{k+0.5}{m}, \xi_3 = \frac{k+1}{m}.$$

In the above definition integer $m = 2^j$, $j = 0,1, \dots, J$, indicates the level of the wavelet and integer $k = 0,1, \dots, m-1$ is the translation parameter. Maximum level of resolution is J. The index i in Eq. (3) is calculated using the formula $i = m + k + 1$. In case of minimal values $m = 1, k = 0$, we have $i = 2$.The maximal value of i is $i = 2M = 2^{J+1}$.

Any square integrable function $u(t)$ defined on [0,1) can be approximated using Haar wavelet series as

$$u(t) \simeq \sum_{i=1}^{N} \lambda_i h_i(t). \tag{4}$$

Let $p_{i,1}(x)$ denotes the integral of Haar function as defined below.

$$p_{i,1}(t) = \int_0^t h_i(z)dz = \begin{cases} t-\alpha & t \in [\xi_1, \xi_2), \\ \gamma - t & t \in [\xi_2, \xi_3), \\ 0 & \text{otherwise}. \end{cases} \tag{5}$$

3. NUMERICAL PROCEDURE

In this section, proposed numerical method will be developed to find numerical solution of linear as well as nonlinear fractional delay differential equations using Haar wavelet collocation method. For Haar wavelet collocation method the highest derivative involved is approximated using Haar wavelet in the following way

$$\dot{u}(t) = \sum_{i=1}^{N} \lambda_i h_i(t). \tag{6}$$

The approximate expression for the solution $u(t)$ is obtained by integrating the above equation and thus we have

$$u(t) = u_0 + \sum_{i=1}^{N} \lambda_i p_{i,1}(t), \tag{7}$$

where $u_0 = u(0)$.

Linear Case

Consider the initial value problem for a linear fractional delay differential equation with finite delay $\tau > 0$, given in Eq. (1). Applying the Caputo derivative, we obtain

$$\frac{1}{\Gamma(n-\alpha)}\int_0^t u^{(n)}(\tau)(t-\tau)^{n-\alpha-1}d\tau = a\,u(t-\tau) + b\,u(t) + f(t) \quad (8)$$

We will describe the method for $n = 1$ only. For other values of n, a similar procedure can be adopted. For $n = 1$, the above equation becomes

$$\frac{1}{\Gamma(n-\alpha)}\int_0^t \dot{u}(\tau)(t-\tau)^{-\alpha}d\tau = a\,u(t-\tau) + b\,u(t) + f(t). \quad (9)$$

Next applying the Haar wavelet approximations we obtain

$$\sum_{i=1}^{N}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t h_i(\tau)(t-\tau)^{-\alpha}d\tau - b\,p_{i,1}(t)\right)\lambda_i = a\,u(t-\tau) + b\,u_0 + f(t). \quad (10)$$

Discretizing the above equation, we obtain

$$\sum_{i=1}^{N}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^{t_j} h_i(\tau)\left(t_j-\tau\right)^{-\alpha}d\tau - b\,p_{i,1}\left(t_j\right)\right)\lambda_i = a\,u\left(t_j-\tau\right) + b\,u_0 + f\left(t_j\right),$$

$j = 1,2,\ldots,N$, (11)

where $t_j, j = 1,2,\cdots,N$ are the collocation points defined as:

$$t_j = \frac{j-0.5}{N}, j = 1,2,\cdots,N. \quad (12)$$

The following notation is introduced:

$$G_{ji} = \frac{1}{\Gamma(n-\alpha)}\int_0^{t_j} h_i(\tau)\left(t_j-\tau\right)^{-\alpha}d\tau - b\,p_{i,1}\left(t_j\right), i,j = 1,2,\cdots N. \quad (13)$$

With this notation Eq. (11) can be written in matrix form as:

$$\boldsymbol{G\lambda} = \boldsymbol{B},$$

where $\boldsymbol{G} = [G_{ji}]_{N\times N}, \boldsymbol{\lambda} = [\lambda_i]_{N\times 1}, \boldsymbol{B} = [B_i]_{N\times 1}$. The entries in the matrix G are calculated as [10]:

$$G_{ji} = 0, 0 \le t_j < \xi_1,$$

$$G_{ji} = \frac{(t_j-\xi_1)^{1-\alpha}}{\Gamma(1-\alpha)(1-\alpha)} - b\,p_{i,1}\left(t_j\right), \quad \xi_1 \le t_j < \xi_2,$$

$$G_{ji} = \frac{\left((t_j-\xi_1)^{1-\alpha} - 2(t_j-\xi_2)^{1-\alpha}\right)}{\Gamma(1-\alpha)(1-\alpha)} - b\,p_{i,1}\left(t_j\right), \xi_2 \le t_j < \xi_3,$$

and

$$G_{ji} = \frac{\left((t_j-\xi_1)^{1-\alpha} - 2(t_j-\xi_2)^{1-\alpha} - (t_j-\xi_3)^{1-\alpha}\right)}{\Gamma(1-\alpha)(1-\alpha)} - b\,p_{i,1}(t_j), \xi_3 \le t_j < 1,$$

whereas the entries in the matrix $\boldsymbol{B}$ are given below:

$$B_j = \begin{cases} a\,\varphi\left(t_j-\tau\right) + b\,u(t_j) + f\left(t_j\right) & t_j < 0 \\ a\,u\left(t_j-\tau\right) + b\,u(t_j) + f\left(t_j\right) & t_j > 0 \end{cases}$$

Hence the unknowns $\lambda_i, i = 1,2,\cdots,N$ are calculated as

$$\boldsymbol{\lambda} = \boldsymbol{G}^{-1}\boldsymbol{B}$$

The approximate solution at the collocation points is finally calculated by substituting $\lambda_i, i = 1,2,\cdots,N$ in Eq. (7).

Nonlinear Case

We consider the nonlinear fractional delay differential equation in the following form:

$$D^{\alpha}u(t) = f\big(t, u(t), u(t-\tau)\big).$$

By applying a similar procedure discussed for the linear case we obtain

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t_j-\tau)^{-\alpha}\sum_{i=1}^{N}\lambda_i h_i(\tau)\,d\tau$$
$$= f\left(t, u_0 + \sum_{i=1}^{N}\lambda_i p_{i,1}(t), \varphi(t)\right)$$

Substituting the collocation points we obtain a nonlinear system which can be solved using Newton's method or Broyden's method.

4. NUMERICAL EXPERIMENTS

In this section three test problems are considered to illustrate the accuracy and efficiency of the proposed method.

Test Problem 1. Consider the following linear fractional delay differential equation [8]:

$$D^{\frac{1}{2}}u(t) = u(t-1) - u(t) + 2t - 1 + \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}},$$
$$u(t) = t^2, t \in [-1,0].$$

The exact solution of the above problem is $u(t) = t^2$.

Test Problem 2. Consider the following linear fractional delay differential equation [8]:

$$D^{\frac{1}{2}}u(t) = u(t-1) - t$$
$$u(t) = t, t \in [-1,0].$$

The exact solution of the above problem is given by

$$u(t) = -\frac{2}{\Gamma\left(\frac{1}{2}\right)}\sqrt{t}.$$

Test Problem 3. Consider the following nonlinear fractional delay differential equation [11]:

$$D^{1.5}u(t) = u(t-0.5) + u^3(t) + \frac{2}{\Gamma(1.5)}t^{0.5} - (t-0.5)^2 - t^6,$$
$$u(t) = t^2, t \in [-0.5,0],$$

subject to the boundary conditions

$$u(0) = 0, u(1) = 1.$$

The exact solution of the above problem is $u(t) = t^2$.

Discussion

Numerical results in terms of maximum absolute errors for all the three test problems are shown in Table 1. It is observed from the table that maximum absolute errors are decreased with the increase in number of collocation points.

Table 1: Maximum absolute errors

N	Test Problem 1	Test Problem 2	Test Problem 3
2	2.2×10^{-2}	1.2×10^{-1}	1.6×10^{-1}
4	7.1×10^{-3}	8.6×10^{-2}	1.5×10^{-1}
8	2.8×10^{-3}	6.1×10^{-2}	1.2×10^{-1}
16	1.1×10^{-3}	4.3×10^{-2}	9.0×10^{-2}
32	4.0×10^{-4}	3.0×10^{-2}	6.5×10^{-2}
64	1.4×10^{-4}	2.1×10^{-2}	4.6×10^{-2}
128	5.2×10^{-5}	1.5×10^{-2}	3.2×10^{-2}
256	1.9×10^{-5}	1.1×10^{-2}	2.3×10^{-2}
512	6.7×10^{-6}	7.6×10^{-3}	—
1024	2.4×10^{-6}	5.4×10^{-3}	—
2048	8.4×10^{-7}	3.8×10^{-3}	—

5. CONCLUSION

A new numerical method is developed using Haar wavelet for the numerical solution of fractional delay differential equations. The numerical results show that the method is efficient and accurate. The performance of the method is equally good for fractional delay differential equations. The method is applicable to both linear and nonlinear problems of fractional delay differential equations.

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ADDITIONAL DATA ABOUT AUTHORS

1. Mr. Imran Aziz is an Associate Professor and Head of Mathematics Department at University of Peshawar, Pakistan.
   Email: imran_aziz@upesh.edu.pk
2. Mr. Rohul Amin is a lecturer and a PhD student at the Department of Math., University of Peshawar.
3. Mr. Jüri Majak is a lead research fellow at the Department of Mech. and Industrial Eng., TUT.
   Email: juri.majak@ttu.ee